Lecture 4

Huaxin Lin

June 9th, 2015,
In this lecture, we will try to settle the following problem:
In this lecture, we will try to settle the following problem: Let $A$ be a unital sub-homogeneous $C^*$-algebra.
In this lecture, we will try to settle the following problem: Let $A$ be a unital sub-homogeneous $C^*$-algebra. Let $\phi, \psi : A \to C$ be two almost multiplicative c.c.p. maps,
In this lecture, we will try to settle the following problem: Let $A$ be a unital sub-homogeneous $C^*$-algebra. Let $\phi, \psi : A \to C$ be two almost multiplicative c.c.p. maps, where $C$ is an 1-dimensional NCCW complex.
In this lecture, we will try to settle the following problem: Let $A$ be a unital sub-homogeneous $C^*$-algebra. Let $\phi, \psi : A \to C$ be two almost multiplicative c.c.p. maps, where $C$ is an 1-dimensional NCCW complex. When are they are approximately unitary equivalent?
In this lecture, we will try to settle the following problem: Let $A$ be a unital sub-homogeneous $C^*$-algebra. Let $\phi, \psi : A \to C$ be two almost multiplicative c.c.p. maps, where $C$ is an 1-dimensional NCCW complex. When are they approximately unitary equivalent?
In this lecture, we will try to settle the following problem: Let $A$ be a unital sub-homogeneous $C^*$-algebra. Let $\phi, \psi : A \to C$ be two almost multiplicative c.c.p. maps, where $C$ is an 1-dimensional NCCW complex. When are they are approximately unitary equivalent? Most advanced results are taken from a joint work with Gong and Niu.
In this lecture, we will try to settle the following problem: Let $A$ be a unital sub-homogeneous $C^*$-algebra. Let $\phi, \psi : A \to C$ be two almost multiplicative c.c.p. maps, where $C$ is an 1-dimensional NCCW complex. When are they approximately unitary equivalent? Most advanced results are taken from a joint work with Gong and Niu. But first we will establish some Bott-map related existence theorems.
Lemma 4.1.

Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT,
Lemma 4.1.

Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup.
Lemma 4.1.

Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers.
Lemma 4.1.

Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers. For any $\delta > 0$, any finite subset $G \subset A$
Lemma 4.1.
Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers. For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N_0, N_1, \ldots, N_k$ and unital homomorphisms $h_j : A \to M_{N_j}$, $j = 1, 2, \ldots, k$ satisfying the following: for any $\kappa \in \text{Hom} \Lambda(K(A), K(K))$, with $|\kappa([1_A])| = J_1$ and $J_0 = \max \{|\kappa(g_i)| : g_i = (i-1 \mathbb{Z}^r : 1 \leq i \leq r) \in \mathbb{Z}_r\}$, there exists a $\delta$-multiplicative contractive completely positive linear map $\Phi : A \to M_{N_0 + \kappa([1_A])}$, such that $[\Phi]|_{\mathcal{P}} = \kappa + [h_1] + [h_2] + \cdots + [h_k]|_{\mathcal{P}}$. (e 0.2)
Lemma 4.1.

Let $A$ be a unital separable amenable residually finite dimensional C*-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers. For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N_0, N_1, ..., N_k$ and unital homomorphisms $h_j : A \rightarrow M_{N_j}$, $j = 1, 2, ..., k$.
Lemma 4.1.
Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers. For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N_0, N_1, \ldots, N_k$ and unital homomorphisms $h_j : A \to M_{N_j}, j = 1, 2, \ldots, k$ satisfying the following: for any $\kappa \in \text{Hom}_\Lambda(K(A), K(\mathbb{C})), \text{ with } |\kappa([1_A])| = J_1$
Lemma 4.1.
Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers. For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N_0, N_1, \ldots, N_k$ and unital homomorphisms $h_j : A \to M_{N_j}$, $j = 1, 2, \ldots, k$ satisfying the following: for any $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$, with $|\kappa([1_A])| = J_1$ and

$$J_0 = \max\{|\kappa(g_i)| : g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r : 1 \leq i \leq r\}, \quad (e \ 0.1)$$
Lemma 4.1.

Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers. For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$ and any finite subset $\mathcal{P} \subset K_0(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N_0, N_1, \ldots, N_k$ and unital homomorphisms $h_j : A \to M_{N_j}$, $j = 1, 2, \ldots, k$ satisfying the following:

for any $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$, with $|\kappa([1_A])| = J_1$ and

$$J_0 = \max \{ |\kappa(g_i)| : g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r : 1 \leq i \leq r \}, \quad (e\ 0.1)$$

there exists a $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Phi : A \to M_{N_0 + \kappa([1_A])}$.
Lemma 4.1.

Let $A$ be a unital separable amenable residually finite dimensional $C^*$-algebra with UCT, let $G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup with $[1_A] \in G$ and let $J_0, J_1 \geq 0$ be integers. For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N_0, N_1, \ldots, N_k$ and unital homomorphisms $h_j : A \to M_{N_j}$, $j = 1, 2, \ldots, k$ satisfying the following: for any $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$, with $|\kappa([1_A])| = J_1$ and

$$J_0 = \max\{|\kappa(g_i)| : g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r : 1 \leq i \leq r\}, \quad (e\, 0.1)$$

there exists a $\delta$-$G$-multiplicative contractive completely positive linear map $\Phi : A \to M_{N_0 + \kappa([1_A])}$, such that

$$[\Phi]|_\mathcal{P} = (\kappa + [h_1] + [h_2] \cdots + [h_k])|_{\mathcal{P}}. \quad (e\, 0.2)$$
Proof.

It follows from 6.1.11 of [Linbook] that,

\[ L_\kappa : A \to M_{n}(\kappa) \quad \text{for integer } n(\kappa) \geq 1 \]

such that

\[ |L_\kappa|_{P} = (\kappa + h_\kappa) |P|, \quad (\ref{eq:3}) \]

where \( h_\kappa : A \to M_{N(\kappa)} \) is a unital homomorphism.

There are only finitely many different \( \kappa |P \) so that (\ref{eq:3}) holds. Say these are given by \( \kappa_1, \kappa_2, \ldots, \kappa_k \).

Set \( h_i = h_{\kappa_i} \), \( i = 1, 2, \ldots, k \).

Let \( N_i = N_{\kappa_i} \), \( i = 1, 2, \ldots, k \).

Note that \( N_i = J_1 + n(\kappa_i) \), if \( \kappa([1_A]) = J_1 \),

and \( N_i = -J_1 + n(\kappa_i) \), if \( \kappa([1_A]) = -J_1 \).

Define \( N_0 = \sum_{i=1}^{k} N_i \).

If \( \kappa = \kappa_i \), define \( \Phi : A \to M_{N_0 + \kappa([1_A])} \) by

\[ \Phi = L_{\kappa_i} + \sum_{j \neq i} h_j. \]

The lemma follows.
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$)
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$G$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P},$$

(e 0.3)
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P},$$

(e 0.3)

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism.
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$G$-multiplicative contractive completely positive linear map $L_\kappa : A \rightarrow M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P},$$  \hspace{1cm} (e 0.3)

where $h_\kappa : A \rightarrow M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P},$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$ so that (??) holds.
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P}, \quad \text{(e0.3)}$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$ so that (??) holds. Say these are given by $\kappa_1, \kappa_2, \ldots, \kappa_k$. 
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$G$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P},$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$ so that (??) holds. Say these are given by $\kappa_1, \kappa_2, ..., \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, ..., k$. 
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P},$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$ so that (??) holds. Say these are given by $\kappa_1, \kappa_2, \ldots, \kappa_k$. Set $h_i = h_{\kappa_i}, i = 1, 2, \ldots, k$. Let $N_i = N_{\kappa_i}, i = 1, 2, \ldots k$. 

Huaxin Lin
Lecture 4
June 9th, 2015
Proof.

It follows from 6.1.11 of [Linbook] that, for each such \( \kappa \), there is a unital \( \delta \)-\( G \)-multiplicative contractive completely positive linear map \( L_\kappa : A \to M_{n(\kappa)} \) (for integer \( n(\kappa) \geq 1 \)) such that

\[
[L_\kappa]|_P = (\kappa + [h_\kappa])|_P, \tag{e 0.3}
\]

where \( h_\kappa : A \to M_{N_\kappa} \) is a unital homomorphism. There are only finitely many different \( \kappa|_P \) so that (??) holds. Say these are given by \( \kappa_1, \kappa_2, \ldots, \kappa_k \). Set \( h_i = h_{\kappa_i}, \ i = 1, 2, \ldots, k \). Let \( N_i = N_{\kappa_i}, \ i = 1, 2, \ldots k \). Note that \( N_i = J_1 + n(\kappa_i) \), if \( \kappa([1_A]) = J_1 \),
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$G$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_P = (\kappa + [h_\kappa])|_P,$$  \hspace{1cm} (e0.3)

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_P$ so that (e0.3) holds. Say these are given by $\kappa_1, \kappa_2, ..., \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, ..., k$. Let $N_i = N_{\kappa_i}$, $i = 1, 2, ... k$. Note that $N_i = J_1 + n(\kappa_i)$, if $\kappa([1_A]) = J_1$, and $N_i = -J_1 + n(\kappa_i)$,
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta\mathcal{G}$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P}, \quad (e\ 0.3)$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$ so that (??) holds. Say these are given by $\kappa_1, \kappa_2, \ldots, \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, \ldots, k$. Let $N_i = N_{\kappa_i}$, $i = 1, 2, \ldots k$. Note that $N_i = J_1 + n(\kappa_i)$, if $\kappa([1_A]) = J_1$, and $N_i = -J_1 + n(\kappa_i)$, if $\kappa([1_A]) = -J_1$. 
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$G$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P}, \quad (\text{e} \ 0.3)$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$ so that (??) holds. Say these are given by $\kappa_1, \kappa_2, \ldots, \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, \ldots, k$. Let $N_i = N_{\kappa_i}$, $i = 1, 2, \ldots k$.

Note that $N_i = J_1 + n(\kappa_i)$, if $\kappa([1_A]) = J_1$, and $N_i = -J_1 + n(\kappa_i)$, if $\kappa([1_A]) = -J_1$. Define

$$N_0 = \sum_{i=1}^{k} N_i.$$
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$G$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_P = (\kappa + [h_\kappa])|_P,$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_P$ so that (e 0.3) holds. Say these are given by $\kappa_1, \kappa_2, ..., \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, ..., k$. Let $N_i = N_{\kappa_i}$, $i = 1, 2, ... k$.

Note that $N_i = J_1 + n(\kappa_i)$, if $\kappa([1_A]) = J_1$, and $N_i = -J_1 + n(\kappa_i)$, if $\kappa([1_A]) = -J_1$. Define

$$N_0 = \sum_{i=1}^{k} N_i.$$

If $\kappa = \kappa_i$, define $\Phi : A \to M_{N_0 + \kappa([1_A])}$ by
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_\mathcal{P} = (\kappa + [h_\kappa])|_\mathcal{P},$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_\mathcal{P}$ so that (??) holds. Say these are given by $\kappa_1, \kappa_2, \ldots, \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, \ldots, k$. Let $N_i = N_{\kappa_i}$, $i = 1, 2, \ldots, k$. Note that $N_i = J_1 + n(\kappa_i)$, if $\kappa([1_A]) = J_1$, and $N_i = -J_1 + n(\kappa_i)$, if $\kappa([1_A]) = -J_1$. Define

$$N_0 = \sum_{i=1}^{k} N_i.$$

If $\kappa = \kappa_i$, define $\Phi : A \to M_{N_0 + \kappa([1_A])}$ by

$$\Phi = L_{\kappa_i} + \sum_{j \neq i} h_j.$$
Proof.

It follows from 6.1.11 of [Linbook] that, for each such $\kappa$, there is a unital $\delta$-$G$-multiplicative contractive completely positive linear map $L_\kappa : A \to M_{n(\kappa)}$ (for integer $n(\kappa) \geq 1$) such that

$$[L_\kappa]|_p = (\kappa + [h_\kappa])|_p,$$

where $h_\kappa : A \to M_{N_\kappa}$ is a unital homomorphism. There are only finitely many different $\kappa|_p$ so that (e0.3) holds. Say these are given by $\kappa_1, \kappa_2, \ldots, \kappa_k$. Set $h_i = h_{\kappa_i}$, $i = 1, 2, \ldots, k$. Let $N_i = N_{\kappa_i}$, $i = 1, 2, \ldots k$. Note that $N_i = J_1 + n(\kappa_i)$, if $\kappa([1_A]) = J_1$, and $N_i = -J_1 + n(\kappa_i)$, if $\kappa([1_A]) = -J_1$. Define

$$N_0 = \sum_{i=1}^{k} N_i.$$

If $\kappa = \kappa_i$, define $\Phi : A \to M_{N_0 + \kappa([1_A])}$ by

$$\Phi = L_{\kappa_i} + \sum_{j \neq i} h_j.$$
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup.
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following:
Lemma 4.2.
Let \( A \) be a unital \( C^* \)-algebra as in 4.1 and let
\[ [1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A) \]
be a finitely generated subgroup. There exists \( \Lambda_i \geq 0, i = 1, 2, \ldots, r \), satisfying the following: For any \( \delta > 0 \), any finite subset \( \mathcal{G} \subset A \), there exist integers \( N(\delta, \mathcal{G}, P, i) \geq 1, i = 1, 2, \ldots, r \), satisfying the following:

Let \( \kappa \in \text{Hom}(\mathbb{Z}^r, \mathbb{Z}^r) \) and \( S_i = \kappa(g_i) \), where \( g_i = (i - 1, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r \), there exists a unital \( \mathcal{G} \)-\( \delta \)-multiplicative contractive completely positive linear map \( \mathcal{L} : A \to M_{N_1} \) and a homomorphism \( \mathcal{h} : A \to M_{N_1} \) such that
\[ [\mathcal{L}]|_{\mathcal{P}} = (\kappa + [\mathcal{h}])|_{\mathcal{P}}, \quad (e^{0.4}) \]
where \( N_1 = \sum_{i=1}^{r} (N(\delta, \mathcal{G}, P, i) \pm \Lambda_i) \cdot |S_i| \).
Lemma 4.2.

Let $A$ be a unital C*-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, 
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $P \subset K(A)$ with $P \cap K_0(A) \subset G$, there exist integers $N(\delta, G, P, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$ and any finite subset $\mathcal{P} \subset \text{K}(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N(\delta, \mathcal{G}, \mathcal{P}, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(\mathcal{K}))$ and $S_i = \kappa(g_i)$,
Lemma 4.2.

Let $A$ be a unital C*-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $P \subset K_0(A)$ with $P \cap K_0(A) \subset G$, there exist integers $N(\delta, G, P, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$ and $S_i = \kappa(g_i)$, where

$$g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r,$$
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}' \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $\mathcal{G} \subset A$ and any finite subset $\mathcal{P} \subset K(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N(\delta, \mathcal{G}, \mathcal{P}, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$ and $S_i = \kappa(g_i)$, where

$$g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}'^r,$$

there exists a unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear map $L : A \to M_{N_1}$
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $P \subset K(A)$ with $P \cap K_0(A) \subset G$, there exist integers $N(\delta, G, P, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$ and $S_i = \kappa(g_i)$, where

$$g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r,$$

there exists a unital $G$-$\delta$-multiplicative contractive completely positive linear map $L : A \rightarrow M_{N_1}$ and a homomorphism $h : A \rightarrow M_{N_1}$ such that
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $P \subset K(A)$ with $P \cap K_0(A) \subset G$, there exist integers $N(\delta, G, P, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$ and $S_i = \kappa(g_i)$, where

$g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$, there exists a unital $G$-$\delta$-multiplicative contractive completely positive linear map $L : A \to M_{N_1}$ and a homomorphism $h : A \to M_{N_1}$ such that

$$[L]|_P = (\kappa + [h])|_P,$$

(e0.4)
Lemma 4.2.

Let $A$ be a unital $C^*$-algebra as in 4.1 and let $[1_A] \in G = \mathbb{Z}^r \oplus \text{Tor}(G) \subset K_0(A)$ be a finitely generated subgroup. There exists $\Lambda_i \geq 0$, $i = 1, 2, \ldots, r$, satisfying the following: For any $\delta > 0$, any finite subset $G \subset A$ and any finite subset $\mathcal{P} \subset K_0(A)$ with $\mathcal{P} \cap K_0(A) \subset G$, there exist integers $N(\delta, G, \mathcal{P}, i) \geq 1$, $i = 1, 2, \ldots, r$, satisfying the following:

Let $\kappa \in \text{Hom}_\Lambda(K(A), K(K))$ and $S_i = \kappa(g_i)$, where

$$g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r,$$

there exists a unital $G$-$\delta$-multiplicative contractive completely positive linear map $L : A \to M_{N_1}$ and a homomorphism $h : A \to M_{N_1}$ such that

$$[L]|_\mathcal{P} = (\kappa + [h])|_\mathcal{P},$$

where $N_1 = \sum_{i=1}^r (N(\delta, G, \mathcal{P}, i) \pm \Lambda_i) \cdot |S_i|.$
Proof: Let $\psi_i^+: G \to \mathbb{Z}$ be a homomorphism defined by $\psi_i^+: (g_i) = 1$, $\psi_j^+(g_j) = 0$, if $j \neq i$, and $\psi_i^+|_{\text{Tor}(G)} = 0$, and let $\psi_i^- : G \to \mathbb{Z}$ be a homomorphism defined by $\psi_i^- : (g_i) = -1$ and $\psi_j^- (g_j) = 0$, if $j \neq i$, and $\psi_i^-|_{\text{Tor}(G)} = 0$, $i = 1, 2, \ldots, r$. Note that $\psi_i^- = -\psi_i^+$, $i = 1, 2, \ldots, r$. Let $\Lambda_i = |\psi_i^+([1_A])|$, $i = 1, 2, \ldots, r$.

Let $\kappa_i^+, \kappa_i^- \in \text{Hom}_{\Lambda}(K(A), K(K))$ be such that $\kappa_i^+|_G = \psi_i^+$ and $\kappa_i^- = \psi_i^-$, $i = 1, 2, \ldots, r$. Let $N_0(i) \geq 1$ (in place of $N_0$) be required by ?? for $\delta$, $G$, $J_0 = 1$ and $J_1 = M_i$. Define $N(\delta, G, P, i) = N_0(i)$, $i = 1, 2, \ldots, r$.

Let $\kappa \in \text{Hom}_{\Lambda}(K(A), K(K))$. Then $\kappa|_G = \sum_{i=1}^r S_i \psi_i^+$, where $S_i = \kappa(g_i)$, $i = 1, 2, \ldots, r$.

By applying 4.1, one obtains $G$-$\delta$-multiplicative contractive completely positive linear maps $L_i^\pm : A \to M_{N_0(i)+\kappa_i^\pm([1_A])}$ and a homomorphism $h_i^\pm : A \to M_{N_0(i)}$ such that

$$[L_i^\pm]|_P = (\kappa_i^\pm + [h_i^\pm])|_P, \quad i = 1, 2, \ldots, r. \quad (e\ 0.5)$$
Define \( L = \sum_{i=1}^{r} L_i^{\pm, |S_i|} \), where \( L^{\pm, |S_i|} : A \to M_{|S_i|N_0(i)} \) defined by

\[
L^{\pm, |S_i|}(a) = \text{diag}(L_i^{\pm}(a), ..., L_i^{\pm}(a))
\]

for all \( a \in A \). One checks that \( L : A \to M_{N_1} \), where

\[
N_1 = \sum_{i=1}^{r} |S_i| (\Lambda_i' + N(\delta, G, P, i)) \quad \text{and} \quad \Lambda_i' = \psi_i^+([1_A]), \text{ if } S_i > 0, \text{ or} \quad \Lambda_i' = -\psi_i^+([1_A]), \text{ if } S_i < 0,
\]

is a unital \( \delta-G \)-multiplicative contractive completely positive linear map and

\[
[L]|_P = (\kappa + [h])|_P
\]

for some homomorphism \( h : A \to M_{N_1} \).
Lemma 4.3.
Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset.
Lemma 4.3.
Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, ...
Lemma 4.3.

Let $A$ be a unital sub-homogeneous C*-algebra and let $\mathcal{P} \subset \mathcal{K}(A)$ be a finite subset. Suppose that $G \subset \mathcal{K}(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. 
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map.
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, ...
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following:
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^1 \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $H \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_{+}^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\},$$

(e 0.6)
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_{+}^{1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $\mathcal{H} \subset A_{+} \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\},$$

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. 

\[ (e \ 0.6) \]
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset \mathcal{K}(A)$ be a finite subset. Suppose that $G \subset \mathcal{K}(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap \mathcal{K}_1(A) = \mathbb{Z}^r \oplus \text{Tor}(\mathcal{K}_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_{+}^{1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_{+} \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

\[ K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\}, \tag{e 0.6} \]

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that

\[ \]
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^q \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\},$$

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that $R \geq N(K + 1)$ and
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\}, \quad \text{(e 0.6)}$$

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that $R \geq N(K + 1)$ and $\text{tr} \circ \phi(h) \geq \Delta(h)$ for all $h \in \mathcal{H}$,

$$\text{(e 0.7)}$$
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A_+^{q+1} \setminus \{0\} \rightarrow (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max \{|\kappa(\beta(g_i))| : 1 \leq i \leq r\}, \quad \text{(e 0.6)}$$

where $g_i = (0, ..., 0, 1, 0, ..., 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$G$-multiplicative contractive completely positive linear map $\phi : A \rightarrow M_R$ such that $R \geq N(K + 1)$ and $\text{tr} \circ \phi(h) \geq \Delta(\hat{h})$ for all $h \in \mathcal{H}$, there exists a unitary $u \in M_R$ such that

$$\left\| \phi(f), u \right\| < \epsilon$$

for all $f \in \mathcal{F}$ and

$$Bott(\phi, u) |P| = \kappa \circ \beta |P|.$$
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = Z^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\},$$

where $g_i = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that $R \geq N(K + 1)$ and $\text{tr} \circ \phi(h) \geq \Delta(\hat{h})$ for all $h \in \mathcal{H}$, there exists a unitary $u \in M_R$ such that

$$\|[\phi(f), u]\| < \epsilon \quad \text{for all } f \in \mathcal{F}$$

and

$$\text{(e 0.7)}$$
Lemma 4.3.

Let $A$ be a unital sub-homogeneous $C^*$-algebra and let $\mathcal{P} \subset K(A)$ be a finite subset. Suppose that $G \subset K(A)$ be the group generated by $\mathcal{P}$, $G_1 = G \cap K_1(A) = \mathbb{Z}^r \oplus \text{Tor}(K_1(A))$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ and let $\Delta : A^+_q \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{H} \subset A_+ \setminus \{0\}$ and an integer $N \geq 1$ satisfying the following: Let $\kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C})$ and put

$$K = \max\{|\kappa(\beta(g_i))| : 1 \leq i \leq r\}, \quad \text{(e 0.6)}$$

where $g_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$. Then for any unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : A \to M_R$ such that $R \geq N(K + 1)$ and $\text{tr} \circ \phi(h) \geq \Delta(\hat{h})$ for all $h \in \mathcal{H}$, there exists a unitary $u \in M_R$ such that

$$\|[\phi(f), u]\| < \epsilon \text{ for all } f \in \mathcal{F} \quad \text{and} \quad \text{Bott}(\phi, u)|_\mathcal{P} = \kappa \circ \beta|_\mathcal{P}. \quad \text{(e 0.7)}$$
Proof: To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ is a subset of the unit ball. Let $\Delta_1 = (1/8)\Delta$ and $\Delta_2 = (1/16)\Delta$.

Let $\epsilon_0 > 0$ and $G_0 \subset A$ be a finite subset satisfy the following: If $\phi' : A \to B$ (for any unital $C^*$-algebra $B$) is a unital $\epsilon_0$-$G_0$-multiplicative contractive completely positive linear map and $u' \in B$ is a unitary such that

$$\|\phi'(g)u' - u'\phi'(g)\| < 4\epsilon_0 \text{ for all } g \in G_0,$$

then $\text{Bott}(\phi', u')|_{\mathcal{P}}$ is well defined. Moreover, if $\phi' : A \to B$ is another unital $\epsilon_0$-$G_0$-multiplicative contractive completely positive linear map then

$$\text{Bott}(\phi', u')|_{\mathcal{P}} = \text{Bott}(\phi'', u'')|_{\mathcal{P}},$$

provided that

$$\|\phi'(g) - \phi''(g)\| < 4\epsilon_0 \text{ and } \|u' - u''\| < 4\epsilon_0 \text{ for all } g \in G_0.$$ (e 0.11)

We may assume that $1_A \in G_0$. Let
\[ \mathcal{G}'_0 = \{ g \otimes f : g \in \mathcal{G}_0 \text{ and } f = \{ 1_{C(\mathbb{T})}, z, z^* \} \}. \]

where \( z \) is the identity function on the unit circle \( \mathbb{T} \). We also assume that if \( \Psi' : A \otimes C(\mathbb{T}) \to C \) (to some unital \( C^* \)-algebra \( C \)) is a \( \mathcal{G}'_0 - \epsilon_0 \)-multiplicative contractive completely positive linear map, then there exist a unitary \( u' \in C \) such that

\[ \| \Psi'(1 \otimes z) - u' \| < 4\epsilon_0. \]  

Without loss of generality, we may assume that \( \mathcal{G}_0 \) is in the unital ball of \( A \). Let \( \epsilon_1 = \min\{\epsilon/64, \epsilon_0/512\} \) and \( \mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_0 \).

Let \( \mathcal{H}_0 \subset A_+ \setminus \{0\} \) (in place of \( \mathcal{H} \)) be a finite subset and \( L \geq 1 \) be an integer required by ?? for \( \epsilon_1 \) (in place of \( \epsilon \)) and \( \mathcal{F}_1 \) (in place of \( \mathcal{F} \)) as well as \( \Delta_2 \) (in place of \( \Delta \)).

Let \( \mathcal{H}_1 \subset A_1^n \setminus \{0\} \) be finite subsets, \( \mathcal{G}_1 \subset A \) (in place of \( \mathcal{G} \)) be a finite subset, \( \delta_1 > 0 \) (in place of \( \delta \)), \( \mathcal{P}_1 \subset K(A) \) (in place of \( \mathcal{P} \)) be a finite subset, \( \mathcal{H}_2 \subset A_{s.a.} \) be a finite subset and \( 1 > \sigma > 0 \) be required by ?? for \( \epsilon_1 \) (in place of \( \epsilon \)), \( \mathcal{F}_1 \) (in place of \( \mathcal{F} \)) and \( \Delta_1 \). We may assume that \( [1_A] \in \mathcal{P}_2 \), \( \mathcal{H}_2 \) is in the unit ball of \( A \) and \( \mathcal{H}_0 \subset \mathcal{H}_1 \).
Without loss of generality, we may assume that $\delta_1, \sigma < \epsilon_1/16$ and $\mathcal{F}_1 \subset G_1$. Let $\mathcal{P}_2 = \mathcal{P} \cup \mathcal{P}_1$. Suppose that $A$ has irreducible representations of rank $r_1, r_2, \ldots, r_k$. Fix one irreducible representation $\pi_0 : A \to M_{r_1}$. Let $N(p) \geq 1$ (in place of $N(\mathcal{P}_0)$) and $\mathcal{H}_0 \subset A^1_+ \setminus \{0\}$ (in place of $\mathcal{H}$) be a finite subset required by ?? for $\{1_A\}$ (in place of $\mathcal{P}_0$) and $(1/3)\Delta$. Let $G_0 = G \cap K_0(A)$ and write $G_0 = \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \text{Tor}(G_0)$, where

$$\mathbb{Z}^{s_2} \oplus \text{Tor}(G_0) \subset \ker \rho_A.$$ Let $x_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2}$, $j = 1, 2, \ldots, s_2$. Note that $A \otimes C(\mathbb{T}) \in A_s$ and $A \otimes C(\mathbb{T})$ has irreducible representations of rank $r_1, r_2, \ldots, r_k$. Let

$$\bar{r} = \max\{|(\pi_0)_0^*(x_j)| : 0 \leq j \leq s_1 + s_2\}.$$

Let $\mathcal{P}_3 \subset K(A \otimes C(\mathbb{T}))$ be a finite subset set containing $\mathcal{P}_2$, $\{\beta(g_j) : 1 \leq j \leq r\}$ and a finite subset which generates $\beta(\text{Tor}(G_1))$. Choose $\delta_2 > 0$ and finite subset
in $A \otimes C(\mathbb{T})$, where $\mathcal{G}_2 \subset A$ is a finite subset such that, for any unital $\delta_2$-$\mathcal{G}$-multiplicative contractive completely positive linear map

$\Phi' : A \otimes C(\mathbb{T}) \rightarrow C$ (for any unital $C^*$-algebra $C$ with $T(C) \neq \emptyset$), $[\Phi']|_{\mathcal{P}_3}$ is well defined and

$$[\Phi']|_{\text{Tor}(\mathcal{G}_0) \oplus \beta(\text{Tor}(\mathcal{G}))} = 0. \quad (e\,0.13)$$

We may assume $\mathcal{G}_2 \supset \mathcal{G}_1 \cup \mathcal{F}_1$.

Let $\sigma_1 = \min\{\Delta_2(\hat{h}) : h \in \mathcal{H}_1\}$. Note $K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \beta(K_1(A))$ and $K(A \otimes C(\mathbb{T})) = K(A) \oplus \beta(K(A))$. Consider the subgroup of $K_0(A \otimes C(\mathbb{T}))$:

$$\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{s_2} \oplus \mathbb{Z}^r \oplus \text{Tor}(K_0(A) \oplus \beta(\text{Tor}(K_1(A))).$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. Let $N(\delta_3, \mathcal{G}, \mathcal{P}_3, i)$ and $\Lambda_i$, $i = 1, 2, ..., s_1 + s_2 + r$, be required by ?? (for $A \otimes C(\mathbb{T})$). Choose an integer $n_1 \geq N(p)$ such that

$$\frac{(\sum_{i=1}^{s_1+s_2+r} N(\delta_3, \mathcal{G}, \mathcal{P}_3, i) + 1 + \Lambda_i)N(p)}{n_1 - 1} < \min\{\sigma/16, \sigma_1/2\}. \quad (e\,0.14)$$
Choose \( n > n_1 \) such that
\[
\frac{n_1 + 2}{n} < \min\{\sigma/16, \sigma_1/2, 1/(L + 1)\}. \quad (e \, 0.15)
\]

Let \( \epsilon_2 > 0 \) and let \( \mathcal{F}_2 \subset A \) be a finite subset such that \( [\Psi]|_{\mathcal{P}_2} \) is well defined.

Let \( \epsilon_3 = \min\{\epsilon_2/2, \epsilon_1\} \) and \( \mathcal{F}_3 = \mathcal{F}_1 \cup \mathcal{F}_2 \).

Let \( \delta_4 > 0 \) (in place of \( \delta \)), \( \mathcal{G}_3 \subset A \) (in place of \( \mathcal{G} \)) be a finite subset and let \( \mathcal{H}_3 \subset A_+ \setminus \{0\} \) (in place of \( \mathcal{H}_2 \)) required by ?? for \( \epsilon_3 \) (in place of \( \epsilon \)), \( \mathcal{F}_3 \cup \mathcal{H}_1 \) (in place of \( \mathcal{F} \)), \( \delta_3/2 \) (in place of \( \epsilon_0 \)), \( \mathcal{G}_2 \) (in place of \( \mathcal{G}_0 \)), \( \Delta \), \( \mathcal{H}_1 \) (in place of \( \mathcal{H} \)), \( \min\{\sigma/16, \sigma_1/2\} \) (in place of \( \sigma \)) and \( n^2 \) (in place of \( K \)) required by ?? (with \( L_1 = L_2 \)).

Let \( \mathcal{G} = \mathcal{F}_3 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \) and let \( \delta = \min\{\epsilon_3/16, \delta_4, \delta_3/16\} \). Let \( \mathcal{G}_5 = \{g \otimes f : g \in \mathcal{G}_4, \ f \in \{1, z, z^*\}\} \).

Let \( \mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_0 \). Define
\[
\mathcal{N}_0 = (n + 1)N(p)(\sum_{s_1+s_2+r} N(\delta_3, \mathcal{G}_0, \mathcal{P}_3, i) + \Lambda_i + 1) \quad \text{and define} \quad \mathcal{N} = \mathcal{N}_0 + \mathcal{N}_0 \bar{r}.
\]

Fix any \( \kappa \in KK(A \otimes C(\mathbb{T}), \mathbb{C}) \) with
\[
K = \max\{|\kappa(\beta(g_j))| : 1 \leq j \leq r\}.
\]
Let $R > N(K + 1)$. Suppose that $\phi : A \to M_R$ is a unital \(G\)-\(\delta\)-multiplicative contractive completely positive linear map such that

$$\text{tr} \circ \phi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}.$$  

(e 0.16)

Then, by ??, there exists mutually orthogonal projections \(e_0, e_1, e_2, \ldots, e_n \in M_R\) such that \(e_1, e_2, \ldots, e_n\) are equivalent, \(\text{tr}(e_0) < \min\{\sigma/64, \sigma_1/4\}\) and \(e_0 + \sum_{i=1}^n e_i = 1_{M_R}\), and there exists a unital \(\delta_3/2\)-\(G_2\)-multiplicative contractive completely positive linear map \(\psi_0 : A \to e_0 M_R e_0\) and a unital homomorphism \(\psi : A \to e_1 M_R e_1\) such that

$$\|\phi(f) - (\psi_0(f) \oplus \psi(f), \psi(f), \ldots, \psi(f))\| < \epsilon_3 \text{ for all } f \in \mathcal{F}_3 \text{ and (e 0.17)}$$

$$\text{tr} \circ \psi(h) \geq \Delta(\hat{h})/3n \text{ for all } h \in \mathcal{H}_1\text{ (e 0.18)}$$

Let \(\alpha \in Hom_{\Lambda}(K(A \otimes C(\mathbb{T})), K(M_r))\) be define as follows: \(\alpha|_{K(A)} = [\pi_0]\) and \(\alpha|_{\beta(K(A))} = \kappa|_{\beta(K(A))}\). Let

$$\max\{|\kappa \circ \beta(g_i)| : i = 1, 2, \ldots, r, |\pi_0(x_j)| : 1 \leq j \leq s_1 + s_2\} \leq \max\{K, \tilde{r}\}.$$  

Applying ??, we obtain a unital \(\delta_3\)-\(G\)-multiplicative contractive completely positive linear map \(\Psi : A \otimes C(\mathbb{T}) \to M_{N_1}\), where
\[ N'_1 \leq N_1 = \sum_{j=1}^{s_1+s_2+r} N(\delta_3, G_0, P_3, j) + \Lambda_i \max \{ K, \bar{r} \}, \text{ and a homomorphism } H_0 : A \otimes C(\mathbb{T}) \to H_0(1_A) M_{N'_1} H_0(1_A) \text{ such that } \]

\[ [\Psi]|_{P_3} = (\alpha + [H_0])|_{P_3}. \quad (e \, 0.19) \]

In particular, since \([1_A] \in P_2 \subset P_3, \]

\[ \text{rank} \psi(1_A) = r_1 + \text{rank}(H_0). \]

Note that

\[ \frac{N'_1 + N(p)}{R} \leq \frac{N_1 + N(p)}{N(K + 1)} < 1/(n + 1). \quad (e \, 0.20) \]

Let \( R_1 = \text{rank} e_1. \) Then \( R_1 \geq R/(n + 1). \) So, from (??) \( R_1 \geq N_1 + N(p). \)

In other words, \( R_1 - N'_1 \geq N(p). \) Note that

\[ t \circ \psi(\hat{g}) \geq (1/3)\Delta(\hat{g}) \text{ for all } g \in H_0, \]

where \( t \) is the tracial state on \( M_{R_1}. \) By applying ?? to the case that \( \phi = \pi_0 \oplus H_0 \) and \( P_0 = \{[1_A]\}, \) we obtain a unital homomorphism.
Define $\psi'_0 : A \otimes C(\mathbb{T}) \to e_0 M_R e_0$ by $\psi'_0(a \otimes f) = \psi_0(a) \cdot f(1) \cdot e_0$ for all $a \in A$ and $f \in C(\mathbb{T})$, where $1 \in \mathbb{T}$.
Define $\psi' : A \otimes C(\mathbb{T}) \to e_1 M_R e_1$ by $\psi'(a \otimes f) = \psi(a) \cdot f(1) \cdot e_0$ for all $a \in A$ and $f \in C(\mathbb{T})$. Let $E_1 = \text{diag}(e_1, e_2, \ldots, e_{nn_1})$.
Define $L_1 : A \to E_1 M_R E_1$ by

$$L_1(a) = \pi_0(a) \oplus H_0|_A(a) \oplus h_0(a \otimes 1) \oplus (\psi(f), \ldots, \psi(f))$$

for $a \in A$ and define $L_2 : A \to E_1 M_R E_1$ by

$$L_2(a) = \psi(a \otimes 1) \oplus h_0(a \otimes 1) \oplus (\psi(f), \ldots, \psi(f))$$

for $a \in A$. Note that

$$[L_1]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1}$$

$$\text{tr} \circ L_1(h) \geq \Delta_1(\hat{h}), \quad \text{tr} \circ L_2(h) \geq \Delta_1(\hat{h}) \quad \text{for all } h \in \mathcal{H}_1$$

$$|\text{tr} \circ L_1(g) - \text{tr} \circ L_2(g)| < \sigma \quad \text{for all } g \in \mathcal{H}_2.$$

It follows from ?? that there exists a unitary $w_1 \in E_1 M_R E_1$ such that

$$\|\text{ad} w_1 \circ L_2(a) - L_1(a)\| < \epsilon_1 \quad \text{for all } a \in \mathcal{F}_1.$$
Define $E_2 = (e_1 + e_2 + \cdots + e_{n^2})$ and define $\Phi : A \to E_2 M_R E_2$ by

$$\Phi(f)(a) = \text{diag}(\psi(a), \psi(a), \ldots, \psi(a)) \quad \text{for all } a \in A.$$  

Then

$$\text{tr} \circ \Phi(h) \geq \Delta_2(\hat{h}) \quad \text{for all } h \in \mathcal{H}_0$$

By (??), \( \frac{n}{n_1+2} > L + 1 \). By applying ??, we obtain a unitary $w_2 \in E_2 M_R E_2$ and a unital homomorphism $H_1 : A \to (E_2 - E_1) M_R (E_2 - E_1)$ such that

$$\| \text{ad} \, w_2 \circ \text{diag}(L_1(a), H_1(a)) - \Phi(a) \| < \epsilon_1 \quad \text{for all } a \in A.$$ 

Put

$$w = (e_0 \oplus w_1 \oplus (E_2 - E_1))(e_0 \oplus w_2) \in M_R.$$ 

Define $H'_1 : A \otimes C(\mathbb{T}) \to (E_2 - E_1) M_R (E_2 - E_1)$ by

$$H'_1(a \otimes f) = H_1(a) \cdot f(1) \cdot (E_2 - E_1) \quad \text{for all } a \in A \text{ and } f \in C(\mathbb{T}).$$

Define $\Psi_1 : A \to M_R$ by

$$\Psi_1(f) = \psi'_0(f) \oplus \Psi(f) \oplus h_0 \oplus \psi'(f), \ldots, \psi'(f)) \oplus H'_1(f) \quad \text{for all } f \in A \otimes C(\mathbb{T}).$$
It follows from (??), (??) and (??) that

\[ \| \phi(a) - w^* \Psi_1(a \otimes 1)w \| < \epsilon_1 + \epsilon_1 + \epsilon_3 \text{ for all } a \in \mathcal{F}. \]  

(e 0.29)

Now let \( v \in M_R \) be a unitary such that

\[ \| \Psi_1(1 \otimes z) - v \| < 4\epsilon_1. \]  

(e 0.30)

Put \( u = w^*vw \). Then, we estimate that

\[ \| [\phi(a), u] \| < \min\{\epsilon, \epsilon_0\} \text{ for all } a \in \mathcal{F}_1. \]  

(e 0.31)

Moreover, by (??),(??) and (??),

\[ \text{Bott}(\phi, u)|_\mathcal{P} = \kappa \circ \beta|_\mathcal{P}. \]  

(e 0.32)
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras.
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \rightarrow F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by $A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}$. These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $C$. $A$ is said to be minimal, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$. For $t \in (0, 1)$, define $\pi_t : A \rightarrow F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$. If $t = 0$, define $\pi_0 : A \rightarrow \phi_0(F_1) \subset F_2$ by $\pi_0((f, g)) = \phi_0(g)$ for all $(f, g) \in A$. If $t = 1$, define $\pi_1 : A \rightarrow \phi_1(F_1) \subset F_2$ by $\pi_1((f, g)) = \phi_1(g)$ for all $(f, g) \in A$. In what follows, we will call $\pi_t$ as point-evaluation of $A$ at $t$. There is a canonical map $\pi_e : A \rightarrow F_1$ defined by $\pi_e((f, g)) = g$ for all pair $(f, g) \in A$. It is a surjective map.
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$ 

These $C^*$-algebras are called Elliott-Thomsen building block.
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$ 

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $\mathcal{C}$. 


Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$ 

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $C$. $A$ is said to be \textit{minimal}, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$. 
Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$  

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $\mathcal{C}$. $A$ is said to be *minimal*, if $\ker\phi_0 \cap \ker\phi_1 = \{0\}$.

For $t \in (0, 1)$, define $\pi_t : A \to F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$. 
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$  

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $\mathcal{C}$. $A$ is said to be \textit{minimal}, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$.

For $t \in (0, 1)$, define $\pi_t : A \to F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$. If $t = 0$, define $\pi_0 : A \to \phi_0(F_1) \subset F_2$ by $\pi_0((f, g)) = \phi_0(g)$ for all $(f, g) \in A$. 

Huaxin Lin
Lecture 4
June 9th, 2015, 7 / 1
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$ 

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $\mathcal{C}$. $A$ is said to be minimal, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$.

For $t \in (0, 1)$, define $\pi_t : A \to F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$. If $t = 0$, define $\pi_0 : A \to \phi_0(F_1) \subset F_2$ by $\pi_0((f, g)) = \phi_0(g)$ for all $(f, g) \in A$. If $t = 1$, define $\pi_1 : A \to \phi_1(F_1) \subset F_2$ by $\pi_1((f, g)) = \phi_1(g)$ for all $(f, g) \in A$. 

Huaxin Lin

Lecture 4

June 9th, 2015
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0,1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$ 

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $\mathcal{C}$. $A$ is said to be minimal, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$.

For $t \in (0,1)$, define $\pi_t : A \to F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$. If $t = 0$, define $\pi_0 : A \to \phi_0(F_1) \subset F_2$ by $\pi_0((f, g)) = \phi_0(g)$ for all $(f, g) \in A$. If $t = 1$, define $\pi_1 : A \to \phi_1(F_1) \subset F_2$ by $\pi_1((f, g)) = \phi_1(g)$ for all $(f, g) \in A$. In what follows, we will call $\pi_t$ as point-evaluation of $A$ at $t$. 
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1, \phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $\mathcal{C}$. $A$ is said to be minimal, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$.

For $t \in (0, 1)$, define $\pi_t : A \to F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$. If $t = 0$, define $\pi_0 : A \to \phi_0(F_1) \subset F_2$ by $\pi_0((f, g)) = \phi_0(g)$ for all $(f, g) \in A$. If $t = 1$, define $\pi_1 : A \to \phi_1(F_1) \subset F_2$ by $\pi_1((f, g)) = \phi_1(g)$ for all $(f, g) \in A$. In what follows, we will call $\pi_t$ as point-evaluation of $A$ at $t$. There is a canonical map $\pi_e : A \to F_1$ defined by $\pi_e(f, g) = g$ for all pair $(f, g) \in A$. It is a surjective map.
Definition

Let $F_1$ and $F_2$ be two finite dimensional $C^*$-algebras. Suppose that there are two unital homomorphisms $\phi_0, \phi_1 : F_1 \to F_2$. Denote the mapping torus $M_{\phi_1,\phi_2}$ by

$$A = A(F_1, F_2, \phi_0, \phi_1) = \{(f, g) \in C([0,1], F_2) \oplus F_1 : f(0) = \phi_0(g) \text{ and } f(1) = \phi_1(g)\}.$$ 

These $C^*$-algebras are called Elliott-Thomsen building block. The class of all $C^*$-algebras which are finite dimensional or the above form will be denoted by $C$. $A$ is said to be minimal, if $\ker \phi_0 \cap \ker \phi_1 = \{0\}$.

For $t \in (0,1)$, define $\pi_t : A \to F_2$ by $\pi_t((f, g)) = f(t)$ for all $(f, g) \in A$. If $t = 0$, define $\pi_0 : A \to \phi_0(F_1) \subset F_2$ by $\pi_0((f, g)) = \phi_0(g)$ for all $(f, g) \in A$. If $t = 1$, define $\pi_1 : A \to \phi_1(F_1) \subset F_2$ by $\pi_1((f, g)) = \phi_1(g)$ for all $(f, g) \in A$. In what follows, we will call $\pi_t$ as point-evaluation of $A$ at $t$. There is a canonical map $\pi_e : A \to F_1$ defined by $\pi_e(f, g) = g$ for all pair $(f, g) \in A$. It is a surjective map.
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras.
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$
Denote by $D_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $D_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected,
Denote by $D_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $D_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \to C(Z, F)$ is a unital homomorphism, $B \in D_{k-1}$,
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^*$-algebras with the form:

$$A = \{ (f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a) \},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \to C(Z, F)$ is a unital homomorphism, $B \in \mathcal{D}_{k-1}$, where we assume that there is $d_X > 0$ such that, for any $0 < d \leq d_X$, there exists $s_d^* : \overline{X^d} \to Z$ such that
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \to C(Z, F)$ is a unital homomorphism, $B \in \mathcal{D}_{k-1}$, where we assume that there is $d_X > 0$ such that, for any $0 < d \leq d_X$, there exists $s^d_* : X^d \to Z$ such that

$$s^d_*(x) = x \text{ for all } x \in Z \text{ and } \quad (e0.33)$$
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \to C(Z, F)$ is a unital homomorphism, $B \in \mathcal{D}_{k-1}$, where we assume that there is $d_X > 0$ such that, for any $0 < d \leq d_X$, there exists $s^d_* : X^d \to Z$ such that

$$s^d_*(x) = x \text{ for all } x \in Z \text{ and } \lim_{d \to 0} \|f|_Z \circ s^d_* - f|_{X^d}\| = 0 \text{ for all } f \in C(X, F),$$

(e 0.33)
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^\ast$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^\ast$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \to C(Z, F)$ is a unital homomorphism, $B \in \mathcal{D}_{k-1}$, where we assume that there is $d_X > 0$ such that, for any $0 < d \leq d_X$, there exists $s^d_* : X^d \to Z$ such that

$$s^d_*(x) = x \text{ for all } x \in Z \text{ and} \quad (e \text{0.33})$$

$$\lim_{d \to 0} \|f|_Z \circ s^d_* - f|_{X^d}\| = 0 \text{ for all } f \in C(X, F), \quad (e \text{0.34})$$

where $X^d = \{x \in X : \text{dist}(x, Z) < d\}$. 

Huaxin Lin

Lecture 4

June 9th, 2015, 8 / 1
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \rightarrow C(Z, F)$ is a unital homomorphism, $B \in \mathcal{D}_{k-1}$, where we assume that there is $d_X > 0$ such that, for any $0 < d \leq d_X$, there exists $s^d_* : X^d \rightarrow Z$ such that

$$s^d_*(x) = x \quad \text{for all } x \in Z \quad \text{and} \quad (e \, 0.33)$$

$$\lim_{d \rightarrow 0} \|f|_Z \circ s^d_* - f|_{X^d}\| = 0 \quad \text{for all } f \in C(X, F), \quad (e \, 0.34)$$

where $X^d = \{x \in X : \text{dist}(x, Z) < d\}$. We also assume that, for any $0 < d < d_X/2$ and for any $d > \delta > 0$,
Denote by $D_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $D_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \rightarrow C(Z, F)$ is a unital homomorphism, $B \in D_{k-1}$, where we assume that there is $d_X > 0$ such that, for any $0 < d \leq d_X$, there exists $s_d^* : X^d \rightarrow Z$ such that

$$s_d^*(x) = x \text{ for all } x \in Z \text{ and } \lim_{d \rightarrow 0} \|f|_Z \circ s_d^* - f|_{X^d}\| = 0 \text{ for all } f \in C(X, F),$$

where $X^d = \{x \in X : \text{dist}(x, Z) < d\}$. We also assume that, for any $0 < d < d_X/2$ and for any $d > \delta > 0$, there is a homeomorphism $r : X \setminus X^{d-\delta} \rightarrow X \setminus X^d$
Denote by $\mathcal{D}_0$ the class of all finite dimensional $C^*$-algebras. For $k \geq 1$, denote by $\mathcal{D}_k$ the class of all $C^*$-algebras with the form:

$$A = \{(f, a) \in C(X, F) \oplus B : f|_Z = \Gamma(a)\},$$

where $X$ is a connected metric space and $Z \subset X$ is a nonempty proper compact subset such that $X \setminus Z$ is connected, $\Gamma : B \to C(Z, F)$ is a unital homomorphism, $B \in \mathcal{D}_{k-1}$, where we assume that there is $d_X > 0$ such that, for any $0 < d \leq d_X$, there exists $s^d_* : \overline{X^d} \to Z$ such that

$$s^d_*(x) = x \text{ for all } x \in Z \text{ and }$$

$$\lim_{d \to 0} \|f|_Z \circ s^d_* - f|_{\overline{X^d}}\| = 0 \text{ for all } f \in C(X, F),$$

where $X^d = \{x \in X : \text{dist}(x, Z) < d\}$. We also assume that, for any $0 < d < d_X/2$ and for any $d > \delta > 0$, there is a homeomorphism $r : X \setminus X^{d-\delta} \to X \setminus X^d$ such that

$$\text{dist}(r(x), x) < \delta \text{ for all } x \in X \setminus X^{d-\delta}.$$
Examples: $\mathcal{C} \subset D_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$).
Examples: $\mathcal{C} \subset \mathcal{D}_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$,
Examples: $\mathcal{C} \subset \mathcal{D}_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in \mathcal{D}_{d-1}$. 

Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in \mathcal{D}_{d-1}$. 

Note that $C(X, F) \in \mathcal{D}_1$. All theorems stated for $PM_r(C(X))$ so far works for $C^*$-algebras in $A_d$ for all $d \geq 1$. (Gong-L-Niu)
Examples: $\mathcal{C} \subset \mathcal{D}_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in \mathcal{D}_{d-1}$. Suppose that \( \Gamma : B \rightarrow C(\partial(X), F) \) is a unital homomorphism.
Examples: $C \subset D_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in D_{d-1}$. Suppose that $\Gamma : B \to C(\partial(X), F)$ is a unital homomorphism. Define

$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$
Examples: $C \subset D_1$ (with $X = I = [0, 1]$ and $Z = \partial(I) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = I^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in D_{d-1}$. Suppose that $\Gamma : B \to C(\partial(X), F)$ is a unital homomorphism. Define

$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$

Then $A \in D_d$. 
Examples: $C \subset D_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in D_{d-1}$. Suppose that $\Gamma : B \to C(\partial(X), F)$ is a unital homomorphism. Define

$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$ 

Then $A \in D_d$.

Note that $C(Y, F) \in D_1$. 


Examples: $\mathcal{C} \subset \mathcal{D}_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in \mathcal{D}_{d-1}$. Suppose that $\Gamma : B \to C(\partial(X), F)$ is a unital homomorphism. Define

$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$

Then $A \in \mathcal{D}_d$.

Note that $C(Y, F) \in \mathcal{D}_1$. 
Examples: $C \subset D_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in D_{d-1}$. Suppose that $\Gamma : B \to C(\partial(X), F)$ is a unital homomorphism. Define

$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$ 

Then $A \in D_d$.

Note that $C(Y, F) \in D_1$.

All theorems stated for $PM_r(C(X))P$ so far
Examples: $C \subset D_1$ (with $X = \mathbb{I} = [0, 1]$ and $Z = \partial(\mathbb{I}) = \{0\} \cup \{1\}$). Let $d \geq 1$, $X = \mathbb{I}^d$, the $d$-dimensional disk, $Z = \partial(X)$, $F$ be a finite dimensional $C^*$-algebra and let $B \in D_{d-1}$. Suppose that $\Gamma : B \to C(\partial(X), F)$ is a unital homomorphism. Define

$$A = \{(f, b) \in C(X, F) \oplus B : f|_{\partial(X)} = \Gamma(b)\}.$$ 

Then $A \in D_d$. 

Note that $C(Y, F) \in D_1$. 

All theorems stated for $PM_r(C(X))P$ so far works for $C^*$-algebras in $A_d$ for all $d \geq 1$. (Gong-L-Niu)
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. 

Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q, 1 + \{0\} \to (0, 1)$ be an order preserving map.

There exists a finite subset $H_1 \subset A^1 + \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$, a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A$, a finite subset $U \subset U(M_k + 1(A)) / CU(M_k + 1(A))$ (k depends on A) for which $U \subset P$, and $N \in \mathbb{N}$ satisfying the following:

For any unital $G$-\(\delta\)-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that $|\phi|_P = |\psi|_P$, 

$$\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a),$$

for all $\tau \in T(C)$, $a \in H_1$,

$$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1,$$

for all $a \in H_2$,

and $\text{dist}(\phi^\|, \psi^\|) < \gamma_2$, for all $u \in U$,

there exists a unitary $W \in C \otimes M_N$ such that 

$$\|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon,$$

for all $f \in F$. 

(e 0.36) 

(e 0.37) 

(e 0.38) 

(e 0.39)
Theorem 4.4. Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number. There exists a finite subset $H_1 \subset A_{q+1}^+ \cup \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$, a finite subset $P \subset \mathcal{K}(A)$, a finite subset $H_2 \subset A$, a finite subset $U \subset U(M_{k+1}(A))/C(U(M_{k+1}(A)))$ (k depends on $A$), for which $U \subset P$, and $N \in \mathbb{N}$ satisfying the following: For any unital $G$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that $|\phi|_P = |\psi|_P$, $\tau(\phi(a)) \geq \Delta(a)$, $\tau(\psi(a)) \geq \Delta(a)$, for all $\tau \in \mathcal{T}(C)$, $a \in H_1$, $(e^{0.36})$ $|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1$, for all $a \in H_2$, $(e^{0.37})$ and $\text{dist}(\phi^\tau(u), \psi^\tau(u)) < \gamma_2$, for all $u \in U$, $(e^{0.38})$ there exists a unitary $W \in C \otimes M_N$ such that $\|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon$, for all $f \in F$. $(e^{0.40})$
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_+ \setminus \{0\} \rightarrow (0,1)$ be an order preserving map.

There exists a finite subset $H_1 \subset A_1 + \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$ and a finite subset $P \subset \mathcal{K}(A)$, a finite subset $H_2 \subset A$, a finite subset $U \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ (for $k$ depends on $A$) for which $U \subset P$, and $N \in \mathbb{N}$ satisfying the following:

1. For any unital $G$-multiplicative contractive completely positive linear maps $\phi, \psi : A \rightarrow C$ for some $C \in C$ such that $\|\phi\|_P = \|\psi\|_P$, (e 0.36)

   \[ \tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \quad a \in H_1, \]

2. (e 0.37)

   \[ |\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in H_2, \]

3. (e 0.38)

   \[ \text{dist}(\phi^\|u\|, \psi^\|u\|) < \gamma_2, \quad \text{for all } u \in U, \]

4. (e 0.39)

   \[ \exists \text{ a unitary } W \in C \otimes M_N \text{ such that } \|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon, \quad \text{for all } f \in F. \]
Theorem 4.4.

Let \( A \in \mathcal{D}_d \) for some integer \( d \geq 1 \). Let \( \mathcal{F} \subset A \), let \( \epsilon > 0 \) be a positive number and let \( \Delta : A^{q,1}_+ \setminus \{0\} \rightarrow (0, 1) \) be an order preserving map. There exists a finite subset \( \mathcal{H}_1 \subset A^1_+ \setminus \{0\} \),
Theorem 4.4.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_{+}^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_{+}^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, ...
Theorem 4.4.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_+^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$.
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_+^{q;1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $H_1 \subset A_+^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$ and a finite subset $P \subset K(A)$,
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A)) / CU(M_{k+1}(A))$ (k depends on A) for which $\mathcal{U} \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following:

For any unital $G$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that $\|\phi\|_{\mathcal{P}} = \|\psi\|_{\mathcal{P}}$, $\tau(\phi(a)) \geq \Delta(a)$, $\tau(\psi(a)) \geq \Delta(a)$, for all $\tau \in T(C)$, $a \in \mathcal{H}_1$, (e 0.36)

\[
\tau(\phi(a)) - \tau(\psi(a)) < \gamma_1,
\]

for all $a \in \mathcal{H}_2$, (e 0.37)

\[
\|\tau \circ \phi(a) - \tau \circ \psi(a)\| < \epsilon,
\]

for all $a \in \mathcal{U}$, (e 0.38)

\[
\|\phi^\sharp(u) - \psi^\sharp(u)\| < \gamma_2,
\]

for all $u \in \mathcal{U}$, (e 0.39)

there exists a unitary $W \in C \otimes M_N$ such that $\|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon$, for all $f \in \mathcal{F}$.
Theorem 4.4.
Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map.
There exists a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$).
Theorem 4.4.
Let $A \in D_d$ for some integer $d \geq 1$. Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_{+}^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_{+}^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$,
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following:

\[(e 0.36)\]
Theorem 4.4. Let $A \in D_d$ for some integer $d \geq 1$. Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_{+}^{1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $H_1 \subset A_{+}^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$ and a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A$, a finite subset $U \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[U] \subset P$, and $N \in \mathbb{N}$ satisfying the following: For any unital $G$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_1^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset \mathcal{K}(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that
Theorem 4.4.

Let \( A \in \mathcal{D}_d \) for some integer \( d \geq 1 \). Let \( \mathcal{F} \subset A \), let \( \epsilon > 0 \) be a positive number and let \( \Delta : A^q_+^{1,0} \rightarrow (0, 1) \) be an order preserving map. There exists a finite subset \( \mathcal{H}_1 \subset A^1_+ \setminus \{0\} \), \( \gamma_1 > 0 \), \( \gamma_2 > 0 \), \( \delta > 0 \), a finite subset \( \mathcal{G} \subset A \) and a finite subset \( \mathcal{P} \subset K(A) \), a finite subset \( \mathcal{H}_2 \subset A \), a finite subset \( \mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A)) \) (\( k \) depends on \( A \)) for which \( \mathcal{[U]} \subset \mathcal{P} \), and \( N \in \mathbb{N} \) satisfying the following: For any unital \( \mathcal{G} \)-\( \delta \)-multiplicative contractive completely positive linear maps \( \phi, \psi : A \rightarrow C \) for some \( C \in C \) such that

\[
[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}},
\]
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $F \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $G \subset A$ and a finite subset $P \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $U \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[U] \subset P$, and $N \in \mathbb{N}$ satisfying the following: For any unital $G$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that

$$[[\phi]]_P = [[\psi]]_P,$$  (e 0.36)

$$\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \ a \in \mathcal{H}_1,$$
Theorem 4.4.
Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_1 \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \quad (e\ 0.36)$$

$$\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \ a \in \mathcal{H}_1, \quad (e\ 0.37)$$

$$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2,$$
Theorem 4.4.

Let $A \in \mathcal{D}_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_+^1 \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in \mathcal{C}$ such that

$[\phi]|_\mathcal{P} = [\psi]|_\mathcal{P}$, \hspace{1cm} (e 0.36)

$\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \quad a \in \mathcal{H}_1,$ \hspace{1cm} (e 0.37)

$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2,$ \hspace{1cm} (e 0.38)

and $\text{dist}(\phi^\dagger(u), \psi^\dagger(u)) < \gamma_2$, for all $u \in \mathcal{U}$,
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_{+}^{q,1} \setminus \{0\} \rightarrow (0,1)$ be an order preserving map.

There exists a finite subset $\mathcal{H}_1 \subset A_{+}^{1} \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ ($k$ depends on $A$) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-\(\delta\)-multiplicative contractive completely positive linear maps $\phi, \psi : A \rightarrow C$ for some $C \in C$ such that

$$[\phi]|_\mathcal{P} = [\psi]|_\mathcal{P},$$

$$\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \ a \in \mathcal{H}_1,$$

$$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2,$$

and $\text{dist}(\phi^+(u), \psi^+(u)) < \gamma_2$, for all $u \in \mathcal{U}$,

there exists a unitary $W \in C \otimes M_N$ such that
Theorem 4.4.

Let $A \in D_d$ for some integer $d \geq 1$. Let $\mathcal{F} \subset A$, let $\epsilon > 0$ be a positive number and let $\Delta : A_+^{q,1} \setminus \{0\} \to (0, 1)$ be an order preserving map. There exists a finite subset $\mathcal{H}_1 \subset A_+^1 \setminus \{0\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A$, a finite subset $\mathcal{U} \subset U(M_{k+1}(A))/CU(M_{k+1}(A))$ (k depends on A) for which $[\mathcal{U}] \subset \mathcal{P}$, and $N \in \mathbb{N}$ satisfying the following: For any unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps $\phi, \psi : A \to C$ for some $C \in C$ such that

$$[\phi]|_\mathcal{P} = [\psi]|_\mathcal{P}, \quad (e \ 0.36)$$

$$\tau(\phi(a)) \geq \Delta(a), \quad \tau(\psi(a)) \geq \Delta(a), \quad \text{for all } \tau \in T(C), \ a \in \mathcal{H}_1, \quad (e \ 0.37)$$

$$|\tau \circ \phi(a) - \tau \circ \psi(a)| < \gamma_1, \quad \text{for all } a \in \mathcal{H}_2, \quad (e \ 0.38)$$

and

$$\text{dist}(\phi^\dagger(u), \psi^\dagger(u)) < \gamma_2, \quad \text{for all } u \in \mathcal{U}, \quad (e \ 0.39)$$

there exists a unitary $W \in C \otimes M_N$ such that

$$\|W(\phi(f) \otimes 1_{M_N})W^* - (\psi(f) \otimes 1_{M_N})\| < \epsilon, \quad \text{for all } f \in \mathcal{F}. \quad (e \ 0.40)$$
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$.

Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition so that

$$\pi_{t_i} \circ \phi(g) \approx \pi_{t_i'} \circ \phi(g)$$

and

$$\pi_{t_i} \circ \psi(g) \approx \pi_{t_i'} \circ \psi(g).$$

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$, such that

$$w_i \pi_{t_i} \circ \phi(g) w_i^* \approx \pi_{t_i} \circ \psi(g).$$

We may also assume that there is a unitary $w_e \in F_1$ such that

$$h_0(w_e) = w_0$$

and

$$h_1(w_e) = w_n.$$

Note that

$$(w_i^* w_{i+1} w_i + 1) \pi_{t_i} \circ \phi(g) (w_i w_i^* w_{i+1} w_i + 1) \approx w_i^* w_{i+1} \pi_{t_i+1} \circ \psi(g) w_{i+1} \approx \phi(i+1) \circ \phi(g) \approx \phi(i) \circ \phi(g).$$

We need to apply the Homotopy Lemma.
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. 
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g)$$

and

We need to apply the Homotopy Lemma.
Idea of the proof:

Let \( C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1 \). Let \( 0 = t_0 < t_1 < \cdots < t_n = 1 \)

be a partition so that

\[
\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \text{ and } \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g)
\]  
(e 0.41)
Idea of the proof:

Let \( C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1 \). Let

\[
0 = t_0 < t_1 < \cdots < t_n = 1
\]

be a partition so that

\[
\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g)
\]

(e 0.41)

for all \( g \in G \), provided \( t, t' \in [t_{i-1}, t_i] \), \( i = 1, 2, \ldots, n \).
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e \ 0.41)$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, 

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e \ 0.41)$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$. 

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$,
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad \text{(e 0.41)}$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$. 

---
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e \ 0.41)$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$,
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \text{ and } \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e0.41)$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, ..., n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$, such that

$$w_i \pi_{t_i} \circ \phi(g) w_i^* \approx \pi_{t_i} \circ \psi(g). \quad (e0.42)$$
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e\ 0.41)$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$, such that

$$w_i \pi_{t_i} \circ \phi(g)w_i^* \approx \pi_{t_i} \circ \psi(g). \quad (e\ 0.42)$$

We may also assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$. 
Idea of the proof:

Let \( C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1 \). Let
\[
0 = t_0 < t_1 < \cdots < t_n = 1
\]
be a partition so that
\[
\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e\,0.41)
\]
for all \( g \in G \), provided \( t, t' \in [t_{i-1}, t_i], \ i = 1, 2, \ldots, n \).

By applying Theorem 2.1, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \),
\( w_0 \in h_0(F_1) \), if \( i = 0 \), and \( w_1 \in h_1(F_1) \), if \( i = 1 \), such that
\[
w_i \pi_{t_i} \circ \phi(g) w_i^* \approx \pi_{t_i} \circ \psi(g). \quad (e\,0.42)
\]

We may also assume that there is a unitary \( w_e \in F_1 \) such that
\( h_0(w_e) = w_0 \) and \( h_1(w_e) = w_n \).

Note that
\[
(w_{i+1}^* w_i) \pi_{t_i} \circ \phi(g)(w_i^* w_{i+1}) \approx w_{i+1}^* \pi_{t_{i+1}} \circ \psi(g) w_{i+1}
\]
Idea of the proof:

Let \( C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1 \). Let

\[
0 = t_0 < t_1 < \cdots < t_n = 1
\]

be a partition so that

\[
\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \text{ and } \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g)
\]

(e 0.41)

for all \( g \in G \), provided \( t, t' \in [t_{i-1}, t_i] \), \( i = 1, 2, \ldots, n \).

By applying Theorem 2.1, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \), \( w_0 \in h_0(F_1) \), if \( i = 0 \), and \( w_1 \in h_1(F_1) \), if \( i = 1 \), such that

\[
w_i \pi_{t_i} \circ \phi(g) w_i^* \approx \pi_{t_i} \circ \psi(g).
\]

(e 0.42)

We may also assume that there is a unitary \( w_e \in F_1 \) such that

\[
h_0(w_e) = w_0 \text{ and } h_1(w_e) = w_n.
\]

Note that

\[
(w_{i+1}^* w_i) \pi_{t_i} \circ \phi(g) (w_i^* w_{i+1}) \approx w_{i+1}^* \pi_{t_{i+1}} \circ \psi(g) w_{i+1}
\]

\[
\approx \phi_{i+1} \circ \phi(g) \approx \phi_i \circ \phi(g).
\]

(e 0.43)
Idea of the proof:

Let $C = A(F_1, F_2, h_0, h_1) \subset C([0, 1], F_2) \oplus F_1$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\pi_t \circ \phi(g) \approx \pi_{t'} \circ \phi(g) \quad \text{and} \quad \pi_t \circ \psi(g) \approx \pi_{t'} \circ \psi(g) \quad (e\ 0.41)$$

for all $g \in G$, provided $t, t' \in [t_{i-1}, t_i], \ i = 1, 2, ..., n$.

By applying Theorem 2.1, there exists a unitary $w_i \in F_2$, if $0 < i < n$, $w_0 \in h_0(F_1)$, if $i = 0$, and $w_1 \in h_1(F_1)$, if $i = 1$, such that

$$w_i\pi_{t_i} \circ \phi(g)w_i^* \approx \pi_{t_i} \circ \psi(g). \quad (e\ 0.42)$$

We may also assume that there is a unitary $w_e \in F_1$ such that $h_0(w_e) = w_0$ and $h_1(w_e) = w_n$.

Note that

$$(w_{i+1}^* w_i)\pi_{t_i} \circ \phi(g)(w_i^* w_{i+1}) \approx w_{i+1}^* \pi_{t_{i+1}} \circ \psi(g)w_{i+1}$$

$$\approx \phi_{i+1} \circ \phi(g) \approx \phi_i \circ \phi(g). \quad (e\ 0.43)$$

We need to apply the Homotopy Lemma.
Need to change $w_i$ to something $z_i w_i$ to make “bott” element trivial,
Need to change $w_i$ to something $z_i w_i$ to make “bott” element trivial, which is quite demanding.
Need to change $w_i$ to something $z_i w_i$ to make “bott” element trivial, which is quite demanding. In order not to accumulate errors, the condition (??) is used.
Need to change \( w_i \) to something \( z_i w_i \) to make “bott” element trivial, which is quite demanding. In order not to accumulate errors, the condition (??) is used. We also need to take care of “end points”.
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. 

Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one, $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$.

There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence $0 \rightarrow \text{Aff}(T(A))/\rho A(K_0(A)) \rightarrow U(M_k(A))/CU(M_k(A)) \rightarrow K_1(A) \rightarrow 0$.

Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. If $\phi: A \rightarrow B$ is a unital homomorphism then $\phi^\dagger: U(M_k(A))/CU(M_k(A)) \rightarrow U(M_k(B))/CU(M_k(B))$. Slightly modification, if $\phi$ is almost multiplicative, $\phi^\dagger$ can also be defined.
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence $0 \to Aff(T(A))/\rho_A(K_0(A)) \to U(M_k(A))/CU(M_k(A)) \to K_1(A) \to 0$. Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. If $\varphi: A \to B$ is a unital homomorphism then $\varphi^\sharp: U(M_k(A))/CU(M_k(A)) \to U(M_k(B))/CU(M_k(B))$. Slightly modification, if $\varphi$ is almost multiplicative, $\varphi^\sharp$ can also be defined.
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$.
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. 

---

Let $B$ be another unital $C^*$-algebra of stable rank at most $k$. If $\phi: A \rightarrow B$ is a unital homomorphism then $\phi^\#: U(M_k(A))/CU(M_k(A)) \rightarrow U(M_k(B))/CU(M_k(B))$. Slightly modification, if $\phi$ is almost multiplicative, $\phi^\#$ can also be defined.
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. 

Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. 
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. 
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence.
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

$$0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(M_k(A))/CU(M_k(A)) \to K_1(A) \to 0.$$
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\bigcup_{k=1}^{\infty}(U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

$$0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(M_k(A))/CU(M_k(A)) \to K_1(A) \to 0.$$ 

Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. 

Huaxin Lin

Lecture 4

June 9th, 2015, 13 / 1
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $\cup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

$$0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(M_k(A))/CU(M_k(A)) \to K_1(A) \to 0.$$ 

Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. If $\phi : A \to B$ is a unital homomorphism
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even

$$\bigcup_{k=1}^{\infty} (U(M_k(A))/CU(M_k(A))).$$

There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

$$0 \rightarrow \text{Aff}(T(A))/\rho_A(K_0(A)) \rightarrow U(M_k(A))/CU(M_k(A)) \rightarrow K_1(A) \rightarrow 0.$$

Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. If $\phi : A \rightarrow B$ is a unital homomorphism then

$$\phi^\dagger : U(M_k(A))/CU(M_k(A)) \rightarrow U(M_k(B))/CU(M_k(B)).$$
Let $A$ be a unital $C^*$-algebra and let $U(A)$ be the unitary group of $A$. Denote by $CU(A)$ the closure of the commutator subgroup of $U(A)$. When $A$ has stable rank one $CU(A) \subset U_0(A)$. We will consider the group $U(A)/CU(A)$. Or $U(M_k(A))/CU(M_k(A))$. Or even $igcup_{k=1}^\infty (U(M_k(A))/CU(M_k(A)))$. There is a metric on $U(M_k(A))/CU(M_k(A))$. Let us assume that $A$ has stable rank $\leq k$. C. Thomsen, using de la Harp and Skandalis determinant, showed that there is a splitting exact sequence

$$0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \to U(M_k(A))/CU(M_k(A)) \to K_1(A) \to 0.$$  

Let $B$ is another unital $C^*$-algebra of stable rank at most $k$. If $\phi : A \to B$ is a unital homomorphism then $\phi^\dagger : U(M_k(A))/CU(M_k(A)) \to U(M_k(B))/CU(M_k(B))$. Slightly modification, if $\phi$ is almost multiplicative, $\phi^\dagger$ can also be defined.
Let $A$ be a unital $C^*$-algebra and let $C \in C$, where $C = C(F_1, F_2, \varphi_0, \varphi_1)$ is a NCCW. Suppose that $L : A \rightarrow C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \rightarrow F_1$ is a contractive completely positive linear map such that $\varphi_0 \circ L_e = \pi_0 \circ L$ and $\varphi_1 \circ L_e = \pi_1 \circ L$. Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-$G$-multiplicative, then $L_e$ is also $\delta$-$G$-multiplicative.
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW.
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map.

Define $L_e = \pi_e \circ L$.

Then $L_e : A \to F_1$ is a contractive completely positive linear map such that $\phi_0 \circ L_e = \pi_0 \circ L$ and $\phi_1 \circ L_e = \pi_1 \circ L$.

Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-multiplicative, then $L_e$ is also $\delta$-multiplicative.

Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ as defined. Let $L_1, L_2 : A \to C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $F \subset A$ be a subset.

Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$ and $w_1 \in \pi_1(C) \subset F_2$ such that $\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| < \epsilon$ and $\|w_1^* \pi_1 \circ L_1(a) w_1 - \pi_1 \circ L_2(a)\| < \epsilon$ for all $a \in F$.

Then there exists a unitary $u \in F_1$ such that $\|\phi_0(u) \pi_0 \circ L_1(a) \phi_0(u) - \pi_0 \circ L_2(a)\| < \epsilon$ and $\|\phi_1(u) \pi_1 \circ L_1(a) \phi_1(u) - \pi_1 \circ L_2(a)\| < \epsilon$ for all $a \in F$.
**Definition**

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \rightarrow C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. 

Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-multiplicative, then $L_e$ is also $\delta$-multiplicative.
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and}$$

Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-G-multiplicative, then $L_e$ is also $\delta$-G-multiplicative.
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

\[
\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L.
\]

(e 0.44)

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L.$$  \hspace{1cm} (e 0.44)

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative,
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \rightarrow C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \rightarrow F_1$ is a contractive completely positive linear map such that

\[ \phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \]  

(e 0.44)

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative, then $L_e$ is also $\delta$-$\mathcal{G}$-multiplicative.
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \quad (e \, 0.44)$$

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative, then $L_e$ is also $\delta$-$\mathcal{G}$-multiplicative.

Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$,
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L.$$  \hspace{1cm} (e 0.44)

Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-$G$-multiplicative, then $L_e$ is also $\delta$-$G$-multiplicative.

Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined.
**Definition**

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \quad (e \, 0.44)$$

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative, then $L_e$ is also $\delta$-$\mathcal{G}$-multiplicative.

---

**Lemma**

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined. Let $L_1, L_2 : A \to C$ be two unital completely positive linear maps,
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \rightarrow C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \rightarrow F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \quad (e \ 0.44)$$

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative, then $L_e$ is also $\delta$-$\mathcal{G}$-multiplicative.

Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined. Let $L_1, L_2 : A \rightarrow C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a subset.
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \quad (e 0.44)$$

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative, then $L_e$ is also $\delta$-$\mathcal{G}$-multiplicative.

Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined. Let $L_1, L_2 : A \to C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L.$$  \hfill (e 0.44)

Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-$G$-multiplicative, then $L_e$ is also $\delta$-$G$-multiplicative.

Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined. Let $L_1, L_2 : A \to C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$ and $w_1 \in \pi_1(C) \subset F_2$ such that
Definition

Let \( A \) be a unital \( C^* \)-algebra and let \( C \in \mathcal{C} \), where \( C = C(F_1, F_2, \phi_0, \phi_1) \) is a NCCW. Suppose that \( L : A \to C \) is a contractive completely positive linear map. Define \( L_e = \pi_e \circ L \). Then \( L_e : A \to F_1 \) is a contractive completely positive linear map such that

\[
\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \tag{e 0.44}
\]

Moreover, if \( \delta > 0 \) and \( \mathcal{G} \subset A \) and \( L \) is \( \delta - \mathcal{G} \)-multiplicative, then \( L_e \) is also \( \delta - \mathcal{G} \)-multiplicative.

Lemma

Let \( A \) be a unital \( C^* \)-algebra and let \( C \in \mathcal{C} \), where \( C = C(F_1, F_2, \phi_0, \phi_1) \) is a NCCW as defined. Let \( L_1, L_2 : A \to C \) be two unital completely positive linear maps, let \( \epsilon > 0 \) and let \( \mathcal{F} \subset A \) be a subset. Suppose that there is a unitary \( w_0 \in \pi_0(C) \subset F_2 \) and \( w_1 \in \pi_1(C) \subset F_2 \) such that

\[
\| w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a) \| < \epsilon \quad \text{and}
\]

\[
\| w_1^* \pi_1 \circ L_1(a) w_1 - \pi_1 \circ L_2(a) \| < \epsilon.
\]
Definition

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \quad (e\ 0.44)$$

Moreover, if $\delta > 0$ and $G \subset A$ and $L$ is $\delta$-$G$-multiplicative, then $L_e$ is also $\delta$-$G$-multiplicative.

Lemma

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined. Let $L_1, L_2 : A \to C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$ and $w_1 \in \pi_1(C) \subset F_2$ such that

$$\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| < \epsilon \quad \text{and} \quad (e\ 0.45)$$

Huaxin Lin
Lecture 4
June 9th, 2015, 14 / 1
Definition

Let \( A \) be a unital \( C^* \)-algebra and let \( C \in \mathcal{C} \), where \( C = C(F_1, F_2, \phi_0, \phi_1) \) is a NCCW. Suppose that \( L : A \to C \) is a contractive completely positive linear map. Define \( L_e = \pi_e \circ L \). Then \( L_e : A \to F_1 \) is a contractive completely positive linear map such that

\[
\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \tag{e 0.44}
\]

Moreover, if \( \delta > 0 \) and \( \mathcal{G} \subset A \) and \( L \) is \( \delta \)-\( \mathcal{G} \)-multiplicative, then \( L_e \) is also \( \delta \)-\( \mathcal{G} \)-multiplicative.

Lemma

Let \( A \) be a unital \( C^* \)-algebra and let \( C \in \mathcal{C} \), where \( C = C(F_1, F_2, \phi_0, \phi_1) \) is a NCCW as defined. Let \( L_1, L_2 : A \to C \) be two unital completely positive linear maps, let \( \epsilon > 0 \) and let \( \mathcal{F} \subset A \) be a subset. Suppose that there is a unitary \( w_0 \in \pi_0(C) \subset F_2 \) and \( w_1 \in \pi_1(C) \subset F_2 \) such that

\[
\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| < \epsilon \quad \text{and} \quad \|w_1^* \pi_1 \circ L_1(a) w_1 - \pi_1 \circ L_2(a)\| < \epsilon \tag{e 0.45}
\]
**Definition**

Let $A$ be a unital $C^*$-algebra and let $C \in C$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \to C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \to F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L.$$  \hfill (e 0.44)

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative, then $L_e$ is also $\delta$-$\mathcal{G}$-multiplicative.

**Lemma**

Let $A$ be a unital $C^*$-algebra and let $C \in C$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined. Let $L_1, L_2 : A \to C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$ and $w_1 \in \pi_1(C) \subset F_2$ such that

$$\|w_0^* \pi_0 \circ L_1(a) w_0 - \pi_0 \circ L_2(a)\| < \epsilon \quad \text{and} \quad (e 0.45)$$
**Definition**

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW. Suppose that $L : A \rightarrow C$ is a contractive completely positive linear map. Define $L_e = \pi_e \circ L$. Then $L_e : A \rightarrow F_1$ is a contractive completely positive linear map such that

$$\phi_0 \circ L_e = \pi_0 \circ L \quad \text{and} \quad \phi_1 \circ L_e = \pi_1 \circ L. \quad (e \ 0.44)$$

Moreover, if $\delta > 0$ and $\mathcal{G} \subset A$ and $L$ is $\delta$-$\mathcal{G}$-multiplicative, then $L_e$ is also $\delta$-$\mathcal{G}$-multiplicative.

**Lemma**

Let $A$ be a unital $C^*$-algebra and let $C \in \mathcal{C}$, where $C = C(F_1, F_2, \phi_0, \phi_1)$ is a NCCW as defined. Let $L_1, L_2 : A \rightarrow C$ be two unital completely positive linear maps, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. Suppose that there is a unitary $w_0 \in \pi_0(C) \subset F_2$ and $w_1 \in \pi_1(C) \subset F_2$ such that

$$\|w_0^* \pi_0 \circ L_1(a)w_0 - \pi_0 \circ L_2(a)\| < \epsilon \quad \text{and} \quad (e \ 0.45)$$
Proof:

Write $F_1 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$ and $F_2 = M_{r_1} \oplus M_{r_2} \oplus \cdots \oplus M_{r_l}$. We may assume that, $\ker \phi_0 \cap \ker \phi_1 = \{0\}$. We may assume that $\phi_0|_{M_{n_i}}$ is injective, $i = 1, 2, \ldots, k(0)$ with $k(0) \leq k$, $\phi_0|_{M_{n_i}} = 0$ if $i > k(0)$, and $\phi_1|_{M_{n_i}}$ is injective, $i = k(1), k(1) + 1, \ldots, k$ with $k(1) \leq k$, $\phi_1|_{M_{n_i}} = 0$, if $i < k(1)$. Write $F_{1,0} = \bigoplus_{i=1}^{k(0)} M_{n_i}$ and $F_{1,1} = \bigoplus_{j=k(1)}^{k} M_{n_j}$. Note that $k(1) \leq k(0) + 1$, $\phi_0|_{F_{1,0}}$ and $\phi_1|_{F_{1,1}}$ are injective. Note $\phi_0(F_{1,0}) = \phi_0(F_1) = \pi_0(C)$ and $\phi_1(F_{1,1}) = \phi_1(F_1) = \pi_1(C)$. Let $\psi_0 = (\phi_0|_{F_{1,0}})^{-1}$ and $\psi_1 = (\phi_1|_{F_{1,1}})^{-1}$. For each fixed $a \in A$, since $L_i(a) \in C$ ($i = 0, 1$), there are elements

$$g_{a,i} = g_{a,i,1} \oplus g_{a,i,2} \oplus \cdots \oplus g_{a,i,k(0)} \oplus \cdots \oplus g_{a,i,k} \in F_1,$$

such that $\phi_0(g_{a,i}) = \pi_0 \circ L_i(a)$ and $\phi_1(g_{a,i}) = \pi_1 \circ L_i(a)$, $i = 1, 2, \ldots$, where $g_{a,i,j} \in M_{n_j}$, $j = 1, 2, \ldots, k$ and $i = 1, 2$. Note that such $g_{a,i}$ is unique since $\ker \phi_0 \cap \ker \phi_1 = \{0\}$. Since $w_0 \in \pi_0(C) = \phi_0(F_1)$, there is a unitary

$$u_0 = u_{0,1} \oplus u_{0,2} \oplus \cdots \oplus u_{0,k(0)} \oplus \cdots \oplus u_{0,k}$$

such that $\phi_0(u_0) = w_0$. 
Note that the first $k(0)$ components of $u_0$ is uniquely determined by $w_0$ (since $\phi_0$ is injective on this part) and the components after $k(0)$’s components can be chosen arbitrarily (since $\phi_0 = 0$ on this part). Similarly there exist 
\[ u_1 = u_{1,1} \oplus u_{1,2} \oplus \cdots \oplus u_{1,k(1)} \oplus \cdots \oplus u_{1,k} \]
such that $\phi_1(u_1) = w_1$
Now by ?? and ??, we have
\[
\| \phi_0(u_0)^* \phi_0(g_{a,1}^i) \phi_0(u_0) - \phi_0(g_{a,2}^i) \| < \epsilon \quad \text{and} \quad (e 0.49)
\]
\[
\| \phi_1(u_1)^* \phi_1(g_{a,1}^i) \phi_1(u_1) - \phi_1(g_{a,2}^i) \| < \epsilon \quad \text{for all } a \in \mathcal{F}. \quad (e 0.50)
\]
Since $\phi_0$ is injective on $F_1^i$ for $i \leq k(0)$ and $\phi_1$ is injective on $F_1^i$ for $i > k(0)$ (note that we use $k(1) \leq k(0) + 1$), we have
\[
\| (u_{0,i})^* (g_{a,1,i}^i) u_{0,i} - (g_{a,2,i}^i) \| < \epsilon \quad \forall \ i \leq k(0) \quad \text{and} \quad (e 0.51)
\]
\[
\| (u_{1,i})^* (g_{a,1,i}^i) u_{1,i} - (g_{a,2,i}^i) \| < \epsilon \quad \forall \ i > k(0) \quad (e 0.52)
\]
for all $a \in \mathcal{F}$.
Let \( u = u_{0,1} \oplus \cdots \oplus u_{0,k(0)} \oplus u_{1,k(0)+1} \oplus \cdots \oplus u_{1,k} \in F_1 \)—that is for the first \( k(0) \)'s components of \( u \), we use \( u_0 \)'s corresponding components, and for the last \( k - k(0) \) components of \( u \), we use \( u_1 \)'s. From ?? and ??. we have

\[
\| u^* g_{a,1} u - g_{a,2} \| < \epsilon \quad \text{for all } a \in \mathcal{F}.
\]

Apply \( \phi_0 \) and \( \phi_1 \) to the above inequality, we get ?? and ?? as desired.
Proof of Theorem 4.4. There is $n_0$ such that $n_0x = 0$ for all $x \in K_i(A \otimes C(\mathbb{T}))$, $i = 0, 1$. Set $N = n_0!$. Put $\Delta_1$ be defined in ?? for the given $\Delta$.

Let $\mathcal{H}'_1 \subset A_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) for $\epsilon/32$ (in place of $\epsilon$) and $F$ required by ??.

Let $\delta_1 > 0$ (in place of $\delta$), $G_1 \subset A$ (in place of $G$) be a finite subset and let $P_0 \subset K(A)$ (in place of $P$) be a finite subset required by ?? for $\epsilon/32$ (in place of $\epsilon$), $F$ and $\Delta_1$. We may assume that $\delta_1 < \epsilon/32$ and $(2\delta_1, G_1)$ is a $KK$-pair (see the end of ??).

Moreover, we may assume that $\delta_1$ is so small that if $\|uv - vu\| < 3\delta_1$, then the Exel formula

$$\tau(bott_1(u, v)) = \frac{1}{2\pi\sqrt{-1}}(\tau(\log(u^*vuv^*))$$

holds for any pair of unitaries $u$ and $v$ in any unital $C^*$-algebra $C$ with tracial rank zero and any $\tau \in T(C)$ (see Theorem 3.6 of [??]). Moreover if $\|v_1 - v_2\| < 3\delta_1$, then

$$bott_1(u, v_1) = bott_1(u, v_2).$$
Let \( g_1, g_2, ..., g_k(A) \in U(M_m(A)(A)) \) (\( m(A) \geq 1 \) is an integer) be a finite subset such that \( \{g_1, g_2, ..., \bar{g}_k(A)\} \subset J_c(K_1(A)) \) and such that \( \{[g_1], [g_2], ..., [g_k(A)]\} \) forms a set of generators for \( K_1(A) \). Let \( \mathcal{U} = \{\bar{g}_1, \bar{g}_2, ..., \bar{g}_k(A)\} \subset J_c(K_1(A)) \) be a finite subset.

Let \( \mathcal{U}_0 \subset A \) be a finite subset such that

\[
\{g_1, g_2, ..., g_k(A)\} = \{(a_{i,j}) : a_{i,j} \in \mathcal{U}_0\}.
\]

Let \( \delta_u = \min\{1/256m(A)^2, \delta_1/16m(A)^2\} \), \( G_u = \mathcal{F} \cup G_1 \cup \mathcal{U}_0 \), and let \( P_u = P_0 \).

Let \( \delta_2 > 0 \) (in place of \( \delta \) ), let \( G_2 \subset A \) (in place of \( G \) ), and let \( H_2' \subset A_+ \setminus \{0\} \) (in place of \( H \) ) and let \( N_1 \geq 1 \) (in place of \( N \) ) be an integer required by ?? for \( \delta_u \) (in place of \( \epsilon \) ), \( G_u \) (in place of \( \mathcal{F} \) ), \( P_u \) (in place of \( P \) ), \( \Delta \) and with \( \bar{g}_j \) (in place of \( g_j \) ), \( j = 1, 2, ..., k(A) \) (with \( k(A) = r \)).

Let \( d = \min\{\Delta(\hat{h}) : h \in H_2'\} \). Let \( \delta_3 > 0 \) and let \( G_3 \subset A \otimes C(\mathbb{T}) \) be finite subset satisfying the following: For any \( \delta_3 \)-\( G_3 \)-multiplicative contractive completely positive linear map \( L' : A \otimes C(\mathbb{T}) \to C' \) (for any unital \( C^* \)-algebra \( C' \) with \( T(C') \neq \emptyset \)),

\[
|\tau([L](\beta(\bar{g}_j)))| < d/8, \quad j = 1, 2, ..., k(A). \tag{e 0.53}
\]
Without loss of generality, we may assume that

\[ G_3 = \{ g \otimes z : g \in G'_3 \text{ and } z \in \{1, z, z^*\} \}, \]

where \( G'_3 \subset A \) is a finite subset (by choosing a smaller \( \delta_3 \) and large \( G'_3 \)).

Let \( \epsilon'' = \min\{d/27m(A)^2, \delta_u/2, \delta_2/2m(A)^2, \delta_3/2m(A)^2\} \) and let \( \bar{\epsilon}_1 > 0 \) (in place of \( \delta \)) and \( G_4 \subset A \) (in place of \( G \)) be a finite subset required by ?? for \( \epsilon'' \) (in place of \( \epsilon \)) and \( G_u \cup G'_3 \). Put

\[ \epsilon_1 = \min\{\epsilon'_1, \epsilon''_1, \bar{\epsilon}_1\}. \]

Let \( G_5 = G_u \cup G'_3 \cup G_4 \).

Let \( \mathcal{H}'_3 \subset A^+ \) (in place of \( \mathcal{H}_1 \)), \( \delta_4 > 0 \) (in place of \( \delta \)), \( G_6 \subset A \) (in place of \( G \)), \( \mathcal{H}'_4 \subset A_{s.a.} \) (in place of \( \mathcal{H}_2 \)), \( \mathcal{P}_1 \subset K(A) \) (in place of \( \mathcal{P} \)) and \( \sigma_4 > 0 \) (in place of \( \sigma_2 \)) be the finite subsetc and constants required by Theorem ?? \( \epsilon_1/4 \) (in place \( \epsilon \)) and \( G_5 \) (in place of \( \mathcal{F} \)) and \( \Delta \).

Let \( N_2 \geq N_1 \) such that \((k(A) + 1)/N_2 < d/8\). Choose \( \mathcal{H}'_5 \subset A_+ \setminus \{0\} \) and \( \delta_5 > 0 \) and a finite subset \( G_7 \subset A \) such that, for any \( M_m \) and unital \( \delta_5 \)-\( G_7 \)-multiplicative contractive completely positive linear map \( L' : A \to M_m \), if \( \text{tr} \circ L'(h) > 0 \) for all \( h \in \mathcal{H}'_5 \), then \( m \geq N_2((8/d) + 1) \).
Let $\delta = \min\{\epsilon_1/16, \delta_4/4m(A)^2, \delta_5/4m(A)^2\}$, let $\mathcal{G} = \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7$ and let $\mathcal{P} = \mathcal{P}_u \cup \mathcal{P}_1$. Let

$$\mathcal{H}_1 = \mathcal{H}'_1 \cup \mathcal{H}'_2 \cup \mathcal{H}'_3 \cup \mathcal{H}'_4 \cup \mathcal{H}'_6$$

and let $\mathcal{H}_2 = \mathcal{H}'_4$. Let $\gamma_1 = \sigma_4$ and let

$$0 < \gamma_2 < \min\{d/16m(A)^2, \delta_u/9m(A)^2, 1/256m(A)^2\}.$$

Now suppose that $C \in \mathcal{C}$ and $\phi, \psi : A \to C$ be two unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear maps satisfying the assumption for the above given $\Delta, \mathcal{H}_1, \delta, \mathcal{G}, \mathcal{P}, \mathcal{H}_2, \gamma_1, \gamma_2$ and $\mathcal{U}$. Let

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

be a partition so that

$$\|\pi_t \circ \phi(g) - \pi_{t'} \circ \phi(g)\| < \epsilon_1/16 \quad \text{and} \quad \|\pi_t \circ \psi(g) - \pi_{t'} \circ \psi(g)\| < \epsilon_1(\epsilon 0.54)$$

for all $g \in \mathcal{G}$, provided $t, t' \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

We write $C = A(F_1, F_2, h_0, h_1)$, $F_1 = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_{F(1)}}$ and $F_2 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_{F(2)}}$. By the choice of $\mathcal{H}'_5$,

$$n_j \geq N_2(8/d + 1) \quad \text{and} \quad m_s \geq N_2(8/d + 1), \quad 1 \leq j \leq F(2), \quad 1 \leq s \leq F(\epsilon 0.55)$$
By applying Theorem ??, there exists a unitary \( w_i \in F_2 \), if \( 0 < i < n \), \( w_0 \in h_0(F_1) \), if \( i = 0 \), and \( w_1 \in h_1(F_1) \), if \( i = 1 \), such that

\[
\| w_i \pi_{t_i} \circ \phi(g) w_i^* - \pi_{t_i} \circ \psi(g) \| < \epsilon_1/16 \text{ for all } g \in G_5. \quad (e 0.56)
\]

It follows from ?? that we may assume that there is a unitary \( w_e \in F_1 \) such that \( h_0(w_e) = w_0 \) and \( h_1(w_e) = w_n \).

By (??), let \( \omega_j \in M_{m(A)}(C) \) be a unitary such that \( \omega_j \in CU(M_{m(A)}(C)) \) and

\[
\| \langle (\phi \otimes \text{id}_{M_{m(A)}})(g_j^*) \rangle \langle (\psi \otimes \text{id}_{M_{m(A)}})(g_j) \rangle - \omega_j \| < \gamma_2, \quad j = 1, 2, \ldots, k(A). \\
(e 0.57)
\]

Write

\[
\omega_j = \prod_{l=1}^{e(j)} \exp(\sqrt{-1}a_j^{(l)})
\]

for some selfadjoint element \( a_j^{(l)} \in M_{m(A)}(C), \ l = 1, 2, \ldots, e(j), \ j = 1, 2, \ldots, k(A) \). Write

\[
a_j^{(l)} = (a_j^{(l,1)}, a_j^{(l,2)}, \ldots, a_j^{(l,n_{F(2)})}) \text{ and } \omega_j = (\omega_j,1, \omega_j,2, \ldots, \omega_j,F(2))
\]
in $C([0, 1], F_2) = C([0, 1], M_{n_1}) \oplus \cdots \oplus C([0, 1], M_{n_{F(2)}})$, where $\omega_{j,s} = \exp(\sqrt{-1}a_{j}^{(l,s)})$, $s = 1, 2, \ldots, F(2)$. Then

$$\sum_{l=1}^{e(j)} \frac{n_s(t_s \otimes \text{Tr}_m(A)) (a_j^{(l,s)}(t))}{2\pi} \in \mathbb{Z}, \quad t \in [0, 1],$$

where $t_s$ is the normalized trace on $M_{n_s}$, $s = 1, 2, \ldots, F(2)$. In particular,

$$\sum_{l=1}^{e(j)} n_s(t \otimes \text{Tr}_m(A)) (a_j^{(l,s)}(t)) = \sum_{l=1}^{e(j)} n_s(t \otimes \text{Tr}_m(A)) (a_j^{(l,s)}(t')) \text{ for all } t, t'' \in [0, 1].$$

Let $W_i = w_i \otimes \text{id}_{M_m(A)}$, $i = 0, 1, \ldots, n$ and $W_e = w_e \otimes \text{id}_{M_m(F_1)}$. Then

$$\| \pi_i(\langle \phi \otimes \text{id}_{M_m(A)} \rangle (g_j^*)) \rangle W_i (\pi_i(\langle \phi \otimes \text{id}_{M_m(A)} \rangle (g_j) \rangle) W_i^* - \omega_j(t_e) \| \leq 3m(A)^2\epsilon_1 + 2\gamma_2 < 1/32.$$  \tag{e 0.59}

$$\langle \phi_e \otimes \text{id}_{M_m(A)} \rangle (g_j^*) \rangle W_e (\langle \phi_e \otimes \text{id}_{M_m(A)} \rangle (g_j) \rangle) W_e^* - \pi_e(\omega_j) \| < 3m(A)^2\epsilon_1 + 2\gamma_2 < 1/32.$$  \tag{e 0.60}

We also have
It follows from (??) that there exists selfadjoint elements \( b_{i,j} \in M_{m(A)}(F_2) \) such that

\[
\exp(\sqrt{-1} b_{i,j}) = \omega_j(t_i)^*(\pi_i(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j^*) \rangle) W_i(\pi_i(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j) \rangle) W_i^*) \tag{e 0.62}
\]

and \( b_{e,j} \in M_{m(A)}(F_1) \) such that

\[
\exp(\sqrt{-1} b_{e,j}) = \pi_e(\omega_j)^*(\pi_e(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j^*) \rangle) W_e(\pi_e(\langle \phi \otimes \text{id}_{M_{m(A)}}(g_j) \rangle) W_e^*) \tag{e 0.63}
\]

and

\[
\| b_{i,j} \| < 2 \arcsin(3m(A)^2 \epsilon_1/4 + 2\gamma_2), \quad j = 1, 2, \ldots, k(A), \quad i = 0, 1, \ldots, n(\epsilon 0.64)
\]

We write

\[
\begin{align*}
  b_{i,j} &= (b_{i,j}^{(1)}, b_{i,j}^{(2)}, \ldots, b_{i,j}^{F(2)}) \in F_2 \quad \text{and} \\
  b_{e,j} &= (b_{e,j}^{(1)}, b_{e,j}^{(2)}, \ldots, b_{e,j}^{F(1)}) \in F_1. \tag{e 0.65}
\end{align*}
\]

We also have that

\[
\begin{align*}
  h_0(b_{e,j}) &= b_{0,j} \quad \text{and} \quad h_1(b_{e,j}) = b_{n,j}. \tag{e 0.66}
\end{align*}
\]
Note that

\[
(\pi_i(\langle \phi \otimes \text{id}_{M_m(A)}(g_j^*) \rangle)) W_i (\pi_i(\langle \phi \otimes \text{id}_{M_m(A)}(g_j) \rangle)) W_i^* = \pi_i(\omega_j) \exp(\sqrt{-1} b_{ij} e_0)
\]

for \( j = 1, 2, \ldots, k(A) \) and \( i = 0, 1, \ldots, n \), e.

Then,

\[
\frac{n_s}{2\pi} (t_s \otimes \text{Tr}_{M_n(A)})(b_{i,j}^{(s)}) \in \mathbb{Z}, \quad (e \, 0.68)
\]

where \( t_s \) is the normalized trace on \( M_n, s = 1, 2, \ldots, F(2) \), \( j = 1, 2, \ldots, k(A) \), and \( i = 0, 1, \ldots, n \). We also have

\[
\frac{m_s}{2\pi} (t_s \otimes \text{Tr}_{M_m(A)})(b_{e,j}^{(s)}) \in \mathbb{Z} \quad (e \, 0.69)
\]

where \( t_s \) is the normalized trace on \( M_m, s = 1, 2, \ldots, F(1) \), \( j = 1, 2, \ldots, k(A) \). Let

\[
\lambda_{i,j}^{(s)} = \frac{n_s}{2\pi} (t_s \otimes \text{Tr}_{M_n(A)})(b_{i,j}^{(s)}) \in \mathbb{Z},
\]

where \( t_s \) is the normalized trace on \( M_n, s = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, k(A) \) and \( i = 0, 1, 2, \ldots, n \).
Let 
\[ \lambda_{e,j}^{(s)} = \frac{m_s}{2\pi} (t_s \otimes \text{Tr}_{M_m(A)})(b_{e,j}^{(s)}) \in \mathbb{Z} \]
where \( t_s \) is the normalized trace on \( M_{m_s}, s = 1, 2, ..., F(1) \) and \( j = 1, 2, ..., k(A) \). Let
\[
\lambda_{i,j} = (\lambda_{i,j}^{(1)}, \lambda_{i,j}^{(2)}, ..., \lambda_{i,j}^{(F(2)}) \in \mathbb{Z}^{F(2)} \quad \text{and}
\]
\[
\lambda_{e,j} = (\lambda_{e,j}^{(1)}, \lambda_{e,j}^{(2)}, ..., \lambda_{e,j}^{(F(1))}) \in \mathbb{Z}^{F(1)}. \tag{e 0.70}
\]
We have
\[
|\frac{\lambda_{i,j}^{(s)}}{n_s}| < \frac{d}{4}, \quad s = 1, 2, ..., F(2), \quad \text{and} \tag{e 0.71}
\]
\[
|\frac{\lambda_{e,j}^{(s)}}{m_s}| < \frac{d}{4}, \quad s = 1, 2, ..., F(1), \tag{e 0.72}
\]
\( j = 1, 2, ..., k(A), \ i = 0, 1, 2, ..., n. \)
Define \( \alpha_{i}^{(0,1)} : K_1(A) \to \mathbb{Z}^{F(2)} \) by mapping \([g_j]\) to \( \lambda_{i,j} \), \( j = 1, 2, ..., k(A) \) and \( i = 0, 1, 2, ..., n \), and define \( \alpha_{e}^{(0,1)} : K_1(A) \to \mathbb{Z}^{F(1)} \) by mapping \([g_j]\) to
\[ \lambda_{e,j}, \ j = 1, 2, \ldots, k(A). \text{ We write } K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \beta(K_1(A)) \text{ (see ?? for the definition of } \beta). \text{ Define } \alpha_i : K_\ast(A \otimes C(\mathbb{T})) \to K_\ast(F_2) \text{ as follows:} \]

**On** \( K_0(A \otimes C(\mathbb{T})) \), define

\[
\alpha_i|_{K_0(A)} = [\pi_i \circ \phi]|_{K_0(A)}, \quad \alpha_i|_{\beta(K_1(A))} = \alpha_i \circ \beta|_{K_1(A)} = \alpha_i^{(0,1)} \quad (e \ 0.73)
\]

and on \( K_1(A \otimes C(\mathbb{T})) \),

\[
\alpha_i|_{K_1(A \otimes C(\mathbb{T}))} = 0, \quad (e \ 0.74)
\]

\( i = 0, 1, 2, \ldots, n \), and define \( \alpha_e \in \text{Hom}(K_\ast(A \otimes C(\mathbb{T})), K_\ast(F_1)) \), by

\[
\alpha_e|_{K_0(A)} = [\pi_e \circ \phi]|_{K_0(A)}, \quad \alpha_e|_{\beta(K_1(A))} = \alpha_i \circ \beta|_{K_1(A)} = \alpha_e^{(0,1)} \quad (e \ 0.75)
\]

on \( K_0(A \otimes C(\mathbb{T})) \) and \( (\alpha_e)|_{K_1(A \otimes C(\mathbb{T}))} = 0. \) Note that

\[
(h_0)_\ast \circ \alpha_e = \alpha_0 \quad \text{and} \quad (h_1)_\ast \circ \alpha_e = \alpha_n. \quad (e \ 0.76)
\]

Since \( A \otimes C(\mathbb{T}) \) satisfies the UCT, the map \( \alpha_e \) can be lifted to an element of \( KK(A \otimes C(\mathbb{T}), F_1) \) which is still denoted by \( \alpha_e \). Then define

\[
\alpha_0 = \alpha_e \times [h_0] \quad \text{and} \quad \alpha_n = \alpha_e \times [h_1] \quad (e \ 0.77)
\]
in $KK(A \otimes C(T), F_2)$. For $i = 1, \ldots, n - 1$, also pick a lifting of $\alpha_i$ in $KK(A \otimes C(T), F_2)$, and still denote it by $\alpha_i$. We estimate that

$$
\| (w_i^* w_{i+1}) \pi_{t_i} \circ \phi(g) - \pi_{t_i} \circ \phi(g)(w_i^* w_{i+1}) \| < \epsilon_1/4 \text{ for all } g \in G_5,
$$

$i = 0, 1, \ldots, n - 1$. Let $\Lambda_{i,i+1} : C(T) \otimes A \to F_2$ be a unital contractive completely positive linear map given by the pair $w_i^* w_{i+1}$ and $\pi_{t_i} \circ \phi$ (by ??, see 2.8 of [?]). Denote $V_{i,j} = \langle \pi_{t_i} \circ \phi \otimes \id_{M_m(A)}(g_j) \rangle$, $j = 1, 2, \ldots, k(A)$ and $i = 0, 1, 2, \ldots, n - 1$.

Write

$$V_{i,j} = (V_{i,j,1}, V_{i,j,2}, \ldots, V_{i,j,F(2)}) \in F_2, \ j = 1, 2, \ldots, k(A), \ i = 0, 1, 2, \ldots, n.$$

Similarly, write

$$W_i = (W_{i,1}, W_{i,2}, \ldots, W_{i,F(2)}) \in F_2, \ i = 0, 1, 2, \ldots, n.$$

We have

$$
\| W_i V_{i,j} W_i^* V_{i,j} V_{i,j} V_{i+1} W_{i,j} W_{i+1}^* - 1 \| < 1/16 \quad (\text{e } 0.78)
$$

$$
\| W_i V_{i,j} W_i^* V_{i,j} V_{i+1,j} W_{i+1} V_{i+1,j} W_{i+1,j}^* - 1 \| < 1/16 \quad (\text{e } 0.79)
$$
and there is a continuous path $Z(t)$ of unitaries such that $Z(0) = V_{i,j}$ and $Z(1) = V_{i+1,j}$. Since

$$\| V_{i,j} - V_{i+1,j} \| < \delta_1/12, \quad j = 1, 2, \ldots, k(A),$$

we may assume that $\| Z(t) - Z(1) \| < \delta_1/6$ for all $t \in [0, 1]$. We also write

$$Z(t) = (Z_1(t), Z_2(t), \ldots, Z_{F(2)}(t)) \in F_2 \text{ and } t \in [0, 1].$$

We obtain a continuous path

$$W_i V_{i,j} W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^*$$

which is in $CU(M_{nm(A)})$ for all $t \in [0, 1]$ and

$$\| W_i V_{i,j} W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^* - 1 \| < 1/8 \text{ for all } t \in [0, 1].$$

It follows that

$$(1/2\pi \sqrt{-1})(t_s \otimes \text{Tr}_{M_{m(A)}}) \left[ \log(W_{i,s} V_{i,j,s} W_{i,s}^* V_{i,j,s} Z_s(t)^* W_{i+1,s} Z_s(t) W_{i+1,s}^*) \right]$$
is a constant, where \( t_s \) is the normalized trace on \( M_{n_s} \). In particular,

\[
(1/2\pi \sqrt{-1})(t_s \otimes \text{Tr}_{M_{m(A)}})(\log(W_{i,s}V_{i,j,s}^*W_{i,s}W_{i+1,s}V_{i,j,s}W_{i+1,s}^*)). = (1/2\pi \sqrt{-1})(t_s \otimes \text{Tr}_{M_{m(A)}})(\log(W_{i,s}V_{i,j,s}^*W_{i,s}V_{i,j}V_{i+1,j,s}W_{i+1}V_{i,j,s}W_{i+1,s}^*)). 
\]

Also

\[
W_{i} V_{i,j}^* W_{i}^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* = (\omega_j(t_i) \exp(\sqrt{-1}b_{i,j}))^* \omega_j(t_i) \exp(\sqrt{-1}b_{i+1,j}) = \exp(-\sqrt{-1}b_{i,j})\omega_j(t_i)^* \omega_j(t_{i+1}) \exp(\sqrt{-1}b_{i+1,j}). 
\]

Note that, by (??) and (??), for \( t \in [t_i, t_{i+1}] \),

\[
\|\omega_j(t_i)^* \omega_j(t) - 1\| < 3(3\epsilon'_1 + 2\gamma_2) < 3/32, 
\]

\( j = 1, 2, ..., k(A), \ i = 0, 1, ..., n - 1. \)

By Lemma 3.5 of [?],

\[
(t_s \otimes \text{Tr}_{m(A)})(\log(\omega_j,s(t_i)^* \omega_j,s(t_{i+1}))) = 0. 
\]
It follows that (by the Exel formula, using (??), (??) and (??))

\[ t \otimes \text{Tr}_{m(A)}(\text{bott}_1(V_{i,j}, W_i^* W_{i+1})) \]

\[ = \left( \frac{1}{2\pi \sqrt{-1}} \right) (t \otimes \text{Tr}_{m(A)})(\log(V_{i,j} W_i^* W_{i+1} V_{i,j} W_i^* W_i)) \]  

\[ = \left( \frac{1}{2\pi \sqrt{-1}} \right) (t \otimes \text{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_i^* W_i)) \]  

\[ = \left( \frac{1}{2\pi \sqrt{-1}} \right) (t \otimes \text{Tr}_{m(A)})(\log(\exp(-\sqrt{-1}b_{i,j}) \omega_j(t_i)^* \omega_j(t_{i+1}) \exp(\sqrt{-1}b_{i+1,j}))) \]  

\[ = \left( \frac{1}{2\pi \sqrt{-1}} \right) [(t \otimes \text{Tr}_{k(n)})(-\sqrt{-1}b_{i,j}) + (t \otimes \text{Tr}_{k(n)})(\log(\omega_j(t_i)^* \omega_j(t_{i+1}))) \]  

\[ + (t \otimes \text{Tr}_{k(n)})(\sqrt{-1}b_{i,j})] \]  

\[ = \frac{1}{2\pi} (t \otimes \text{Tr}_{k(n)})(-b_{i,j} + b_{i+1,j}) \]  

for all \( t \in T(F_2) \). In other words,

\[ \text{bott}_1(V_{i,j}, W_i^* W_{i+1}) = -\lambda_{i,j} + \lambda_{i+1,j} \]  

\[ \text{Huaxin Lin} \\
\text{Lecture 4} \\
\text{June 9th, 2015, 17 / 1} \]
\( j = 1, 2, \ldots, m(A), \ i = 0, 1, \ldots, n - 1. \)

Consider \( \alpha_0, \ldots, \alpha_n \in KK(A \otimes C(\mathbb{T}), F_2) \) and \( \alpha_e \in KK(A \otimes C(\mathbb{T}), F_1). \)

Note that

\[
|\alpha_i(g_j)| = |\lambda_{i,j}|,
\]

and by (??), one has

\[
m_s, n_j \geq N_2(8/d + 1).
\]

By applying ?? (using (??), among other items), there are unitaries \( z_i \in F_2, \ i = 1, 2, \ldots, n - 1, \) and \( z_e \in F_1 \) such that

\[
\| [z_i, \pi_{t_i} \circ \phi(g)] \| < \delta_u \text{ for all } g \in G_u \quad \text{(e 0.96)}
\]

\[
\text{Bott}(z_i, \pi_{t_i} \circ \phi) = \alpha_i \text{ and } \text{Bott}(z_e, \pi_e \circ \phi) = \alpha_e. \quad \text{(e 0.97)}
\]

Put

\[
z_0 = h_0(z_e) \text{ and } z_n = h_1(z_e).
\]

One verifies (by (??)) that

\[
\text{Bott}(z_0, \pi_{t_0} \circ \phi) = \alpha_0 \text{ and } \text{Bott}(z_n, \pi_{t_n} \circ \phi) = \alpha_n. \quad \text{(e 0.98)}
\]
Let $U_{i,i+1} = z_i(w_i)^* w_{i+1}(z_{i+1})^*$, $i = 0, 1, 2, ..., n - 1$. Then

$$\| [U_{i,i+1}, \pi_t \circ \phi(g)] \| < \min \{ \delta_1, \delta_2 \}, \quad g \in G_u, \ i = 0, 1, 2, ..., n - 1 \quad (e \, 0.99)$$

Moreover, for $i = 0, 1, 2, ..., n - 1$,

$$\text{bott}_1(U_{i,i+1}, \pi_t \circ \phi) = \text{bott}_1(z_i, \pi_t \circ \phi) + \text{bott}_1((w_i^* w_{i+1}, \pi_t \circ \phi)) + \text{bott}_1((z_{i+1}^*, \pi_t \circ \phi)$$

$$= (\lambda_{i,j}) + (-\lambda_{i,j} + \lambda_{i+1,j}) + (-\lambda_{i+1,j})$$

$$= 0.$$  

Note that for any $x \in \bigoplus_{*=0,1} \bigoplus_{k=1}^{\infty} K_* (A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z})$, one has

$$N x = 0. \quad \text{Therefore}$$

$$\text{Bott}((U_{i,i+1}, ..., U_{i,i+1}), (\pi_t \circ \phi, ..., \pi_t \circ \phi))|_P = N \text{Bott}(U_{i,i+1}, \pi_t \circ \phi)|_P = 0,$$

$$i = 0, 1, 2, ..., n - 1. \quad (e \, 0.100)$$

Note that, by the assumption (??),

$$t_s \circ \pi_t \circ \phi(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_1' \quad (e \, 0.101)$$
where $t_s$ is the normalized trace on $M_{n_s}$, $1 \leq s \leq F(2)$.

By applying ??, using (??), (??) and (??), there exists a continuous path of unitaries, \( \{ \tilde{U}_{i,i+1}(t) : t \in [t_i, t_{i+1}] \} \subset F_2 \otimes M_N(\mathbb{C}) \) such that

\[
\tilde{U}_{i,i+1}(t_i) = \text{id}_{F_2 \otimes M_N(\mathbb{C})}, \quad \tilde{U}_{i,i+1}(t_{i+1}) = (z_i w_i^* w_{i+1} z_{i+1}^*) \otimes 1_{M_N(\mathbb{C})},
\]

and

\[
\| \tilde{U}_{i,i+1}(t) \left( \pi_{t_i} \circ \phi(f), \ldots, \phi_{t_i} \circ \phi(f) \right) \tilde{U}_{i,i+1}(t)^* \left( \pi_{t_i} \circ \phi(f), \ldots, \phi_{t_i} \circ \phi(f) \right) \| < \epsilon
\]

for all $f \in F$ and for all $t \in [t_i, t_{i+1}]$. Define $W \in C \otimes M_N$ by

\[
W(t) = (w_i z_i^* \otimes 1_{M_N}) \tilde{U}_{i,i+1}(t) \text{ for all } t \in [t_i, t_{i+1}],
\]

for $i = 0, 1, \ldots, n - 1$. Note that $W(t_i) = w_i z_i^* \otimes 1_{M_N}$, $i = 0, 1, \ldots, n$. Note also that

\[
W(0) = w_0 z_0^* \otimes 1_{M_N} = h_0(w_e z_e^*) \otimes 1_{M_N}
\]

and

\[
W(1) = w_n z_n^* \otimes 1_{M_N} = h_1(w_e z_e^*) \otimes 1_{M_N}.
\]
So $W \in C \otimes M_N$. One then checks that, by (??), (??), (??) and (??), for $t \in [t_i, t_{i+1}]$,

$$\| W(t)((\pi_t \circ \phi)(f) \otimes 1_{M_N})W(t)^* - (\pi_t \circ \psi)(f) \otimes 1_{M_N}\| < \epsilon$$

for all $f \in \mathcal{F}$. 

\begin{align*}
\| W(t)((\pi_t \circ \phi)(f) \otimes 1_{M_N})W(t)^* - W(t)((\pi_{t_i} \circ \phi)(f) \otimes 1_{M_N})W(t_i)^* \| \\
+ \| W(t)(\pi_{t_i} \circ \phi)(f)W(t)^* - W(t_i)(\pi_{t_i} \circ \phi)(f)W(t_i)^* \| \\
+ \| W(t_i)((\pi_{t_i} \circ \phi)(f) \otimes 1_{M_N})W(t_i)^* - (w_i(\pi_{t_i} \circ \phi)(f)w_i^*) \otimes 1_{M_N} \| \\
+ \| w_i(\pi_{t_i} \circ \phi)(f)w_i^* - \pi_{t_i} \circ \psi(f) \| \\
+ \| \pi_{t_i} \circ \psi(f) - \pi_t \circ \phi(f) \| \\
< \epsilon_1/16 + \epsilon/32 + \delta_u + \epsilon_1/16 + \epsilon_1/16 < \epsilon
\end{align*}