In this section, I discuss how to test the Solow model. The notes closely follow the seminal paper by Mankiw, Romer and Weil published in 1992 in the *Quarterly Journal of Economics*. The paper is titled "A Contribution to the Empirics of Economic Growth". The first sentence sets the stage for the focus of the paper: "This paper takes Robert Solow seriously." The authors conclude that...

"the predictions of the Solow model are, to a first approximation, consistent with the evidence."

I will focus on three features of the Solow model:

1. Steady state analysis of the textbook model.
3. Test of the conditional convergence hypothesis.

1 Testing the Textbook Solow Model

The textbook Solow model gives surprisingly simple testable predictions about long-run standards of living. Start with the production function

\[ Y(t) = K(t)^\alpha (A(t)L(t))^{1-\alpha}, \]  

where \( A \) and \( L \) grow at constant exponential rates \( g \) and \( n \). Written in per effective worker terms, the standard law of motion for capital is

\[ \dot{k}(t) = sk(t)^\alpha - (n + g + \delta)k(t), \]

where capital depreciates at rate \( \delta \) and \( s \) is the constant saving rate. Imposing the steady-state condition \( \dot{k}(t) = 0 \) and solving, we get

\[ k^* = \left[ \frac{s}{n + g + \delta} \right]^{1/(1-\alpha)}. \]

Substituting (3) into the intensive-form production function gives

\[ y(t) = k(t)^\alpha = \left[ \frac{s}{n + g + \delta} \right]^{\alpha/(1-\alpha)}. \]
This is output per effective worker evaluated at the steady state. Taking natural logs of both sides gives
\[
\ln y(t) = \frac{\alpha}{1 - \alpha} \ln \left[ \frac{s}{n + g + \delta} \right].
\] (5)

Rearranging, we get
\[
\ln \left[ \frac{Y(t)}{L(t)} \right] = \ln A(0) + gt + \frac{\alpha}{1 - \alpha} \ln(s) - \frac{\alpha}{1 - \alpha} \ln(n + g + \delta).
\] (6)

The empirical model is closed by imposing the assumption that
\[
\ln A(0) = a + \epsilon,
\] (7)
where \( \epsilon \) is a country-specific shock. The final estimating equation is
\[
\ln \left[ \frac{Y_i}{L_i} \right] = a + gt + \frac{\alpha}{1 - \alpha} \ln(s_i) - \frac{\alpha}{1 - \alpha} \ln(n + g + \delta)_i + \epsilon_i.
\] (8)

If capital’s share of income is one-third \( (\alpha = 1/3) \), then the elasticity of output per worker with respect to \( s \) and \( n + g + \delta \) is 0.5 and -0.5, respectively.

The results are shown in Table I. MRW highlight three results that support the Solow model and one that does not.

Support for the Solow Model:

1. Coefficients on saving \( s \) and population growth \( n \) have the correct sign.

2. Coefficients on saving \( s \) and population growth \( n \) are approximately equal in magnitude.

3. Variation in \( s \) and \( n \) can explain 60\% of the variation in output per capita.

Failure of the Solow Model:

1. Coefficients on saving \( s \) and population growth \( n \) are nearly two times too large.

2 Testing the Augmented Solow Model

We now augment the textbook Solow model to include human capital, \( H(t) \). The new production function is
\[
Y(t) = K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta}.
\] (9)
Let $s_k$ and $s_h$ be the saving rate for physical and human capital. The updated laws of motion for capital are

$$\dot{k}(t) = s_k k(t)^\alpha h(t)^\beta - (n + g + \delta) k(t)$$

(10)

and

$$\dot{h}(t) = s_h k(t)^\alpha h(t)^\beta - (n + g + \delta) h(t).$$

(11)

The steady-state values of physical and human capital are

$$k^* = \left[ \frac{s_k^{1-\beta} s_h^\beta}{n + g + \delta} \right]^{1/(1-\alpha-\beta)}$$

(12)

and

$$h^* = \left[ \frac{s_k^\alpha s_h^{1-\alpha}}{n + g + \delta} \right]^{1/(1-\alpha-\beta)}.$$  

(13)

Substituting (12) and (13) into (9), taking logs and simplifying, we get the following estimating equation:

$$\ln(\frac{Y_i}{L_i}) = a + gt + \frac{\alpha}{1 - \alpha - \beta} \ln(s_{k,i}) + \frac{\beta}{1 - \alpha - \beta} \ln(s_{h,i}) - \frac{\alpha + \beta}{1 - \alpha - \beta} \ln(n + g + \delta)_i + \epsilon_i.$$  

(14)

The results are shown in Table II. If $\alpha = \beta = 1/3$, then the coefficients associate with $s_k$, $s_h$, and $(n + g + \delta)$ should be 1, 1, and -2, respectively. There are several results to highlight.

- The impact of investment in physical capital is predicted to be twice as large in the augmented Solow model. More investment in physical capital leads to more income. More income leads to more human capital. More human capital results in more output.
- The implied share parameter of physical capital, $\alpha$, is now approximately 1/3.
- Investment in human capital is statistically significant and has the correct sign.
- The adjusted $R^2$ is higher (0.59 vs. 0.78).

3 Testing the Convergence Hypothesis

The previous analysis assumes that all countries are at their steady states in 1985. This is a questionable assumption. Now consider the possibility that countries are off steady state. The Solow model (textbook or augmented) predicts conditional convergence. After controlling for differences in steady states, income per capita across countries will converge.
As we showed earlier, the income convergence equation can be written as

\[ \ln y(t) - \ln y(0) = (1 - e^{-\lambda t})[\ln y^* - \ln y(0)]. \] (15)

Substituting the expression for \( y^* \):

\[
y^* = \left[ \frac{s_k^{1-\beta} s_h^\beta}{n + g + \delta} \right]^{\alpha/(1-\alpha-\beta)} \left[ \frac{s_k^{\alpha} s_h^{1-\alpha}}{n + g + \delta} \right]^{\beta/(1-\alpha-\beta)}
\]

into equation (15) and re-arranging gives

\[
\ln y(t)_i - \ln y(0)_i = \frac{\alpha(1 - e^{-\lambda t})}{(1 - \alpha - \beta)} \ln(s_{k,i}) + \frac{\beta(1 - e^{-\lambda t})}{(1 - \alpha - \beta)} \ln(s_{h,i}) - \frac{\beta(1 - e^{-\lambda t})}{(1 - \alpha - \beta)} \ln(n + g + \delta)_i - (1 - e^{-\lambda t}) \ln y(0)_i.
\]

The results are shown in Figure 1 and Tables III, IV and V. Several results are worth highlighting.

- There is no evidence of absolute convergence in the larger samples. The estimated coefficient on \( y(0) \) is approximately zero and statistically insignificant. The adjusted \( R^2 \) is near zero.

- The small (\( N = 22 \)) sample of OECD countries shows evidence of absolute convergence. This makes sense because the steady states of the OECD countries are similar.

- There is strong evidence in favor conditional convergence in the textbook and augmented Solow models. This can also be seen in the figures which "partial out" the effects of saving and population growth.

- The rate of convergence is predicted to be \( \lambda = (n + g + \delta)/(1 - \alpha - \beta) \). The estimated value is around 1.5%-2% per year. This implies a half life of over 35 years. Therefore, differences in standards of living can persist for many decades.

\[1\] See equation 17 in the notes "Solow Growth Model (continued)".