1 Ramsey Growth Model

The Ramsey model extends the Solow model to allow for explicitly optimal behavior by firms and households.

1.1 The Basics

Begin by assuming the following:

- There are a large number of identical households with each member supplying one unit of labor.
- For simplicity, the initial amount of labor is set at unity ($L(0) = 1$) and there is no labor growth ($n = 0$).
- Households own the firms.
- Each firm hires labor and rent capital in competitive input markets. Each firm sells its output in a competitive output market.
- Each firm has access to the CRS production function with $A(t)$ growing at rate $g$ and $A(0) = 1$.
- There is no depreciation of capital (i.e., $\delta = 0$).

1.2 Household Behavior

Households are assumed to be infinitely lived and maximize a discounted stream of future utility given by

$$\int_{t=0}^{\infty} e^{-\rho t} u(C(t)) dt$$

by choosing $C(t)$, the control variable, at each point in time. The instantaneous utility function $u[C(t)]$ is assumed to be in the constant elasticity of substitution (CES) class

$$u[C(t)] = \frac{C(t)^{1-\theta} - 1}{1 - \theta}$$

where $\theta = -C u'' / u'$ is called the coefficient of relative risk aversion. Larger $\theta$s indicate more curvature in the utility function and less willingness to substitute consumption intertemporally. $\sigma = 1/\theta$ is commonly referred to as the intertemporal elasticity of substitution. As $\theta \to 0$, the utility function becomes linear in consumption and $\sigma \to \infty$. 

The households’ constraints are given by (i) the flow budget constraint

\[ \dot{a}(t) = w(t) + r(t) a(t) - C(t), \]

where \( a(t) \) is the sole asset, \( w(t) \) is the wage rate and \( r(t) \) is the asset rate of return; (ii) the no Ponzi-game condition

\[ \lim_{t \to \infty} e^{-R(t)} a(t) \geq 0 \]

\( R(t) = \int_{s=0}^{t} r(s) ds \); and (iii) \( a(0) \) given. Solving this continuous-time dynamic problem involves using calculus of variations (no derivations will be provided at this time). Begin by writing down the present-value Hamiltonian

\[ H = e^{-\rho t} [ \frac{C(t)^{1-\theta}}{1-\theta} ] + \lambda(t) [ w + r(t) a(t) - C(t) ] \]

where \( \lambda(t) \), the costate variable, is the present-value shadow price of income. The first-order conditions for maximization are

\[ \frac{\partial H}{\partial C} = 0 \Rightarrow \lambda(t) = e^{-\rho t} C(t)^{-\theta} \quad (1) \]

\[ \dot{\lambda}(t) = -\frac{\partial H}{\partial a} \Rightarrow \dot{\lambda}(t) = -\lambda(t) r(t) \quad (2) \]

and the transversality condition is

\[ \lim_{t \to \infty} \lambda(t) a(t) = 0. \]

Combining (1) and (2), we get

\[ \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}. \quad (3) \]

Equation (3) is known as the Euler equation for consumption.

### 1.3 Firm Behavior

Firm behavior is simpler. Firms choose labor and capital to maximize profits per period

\[ \Pi(t) = F(K(t), A(t)L(t)) - r(t)K(t) - w(t)L(t), \]

which written in its intensive form is

\[ \Pi(t) = e^{\rho t} [ f(k(t)) - r(t)k(t) - w(t)e^{-\rho t} ]. \]
Since firms take \( r(t) \) and \( w(t) \) as given, they will rent capital up to the point that its marginal product equals rental rate or

\[
    r(t) = f'(k(t)).
\]  

(4)

This will result in zero economic profits if labor is also paid its marginal product or

\[
    w(t) = e^{gt}[f(k(t)) - k(t)f'(k(t))].
\]

1.4 Equilibrium Dynamics and Welfare

Equilibrium dynamics for this economy are given by the capital accumulation equation, (3) and (4). Letting \( C(t) = e^{gt}c(t) \) and \( a(t) = e^{gt}k(t) \), along with the appropriate substitutions, the equilibrium for this economy reduces to

\[
    \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta},
\]

\[
    \dot{k}(t) = f(k(t)) - c(t) - gk(t),
\]

along with the transversality condition and \( k(0) \) given. The dynamic properties of this economy have been studied extensively. Figure 2.4 on page 58 of Romer depicts the dynamics in a phase diagram. The primary conclusions are

1. There exists a unique \((c, k)\) combination that produces steady-state \((\dot{k} = \dot{c} = 0)\) growth. This is given by point E in Figure 2.4 of Romer.

2. For a given \( k(0) \), there is a unique \( c(0) \) that will result in a non-divergent path to the steady state. This path is known as the saddle path and this general property is known as saddle-path stability.

3. Since markets are competitive and there are no externalities, the first welfare theorem of economics states that this competitive equilibrium is Pareto optimal (i.e., no agent can be made better without making another worse off). This is also the same outcome that would be reached by a benevolent social planner that treated all agents equally.

4. The steady-state level of consumption per worker in the Ramsey model (sometimes referred to as the modified golden-rule level) is less than the golden-rule level derived from the Solow model. This happens because impatient optimizing agents are willing to trade off a permanently lower level of future consumption for a higher level of consumption today.