Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
I. **10 points** Let $S = \{1, 2, 3\}$ be a sample space and let $\sigma(S)$ its $\sigma$-algebra consisting of all subsets of $S$. Let $P$ be a set function such that $P(\{1\}) = \frac{1}{3}$, $P(\{2\}) = \frac{1}{6}$ and $P(\{3\}) = \alpha$.

1. Find $\alpha$ such that $P$ is a probability function.

2. Compute $P(\{1, 2\})$, $P(\{1, 3\})$ and $P(\{2, 3\})$.

**Solution:**

1. Since $P(S) = 1$ and $P(S) = P(\{1, 2, 3\}) = P(\{1\}) + P(\{2\}) + P(\{3\})$ we get that $1 = \frac{1}{3} + \frac{1}{6} + \alpha \implies \alpha = \frac{1}{2}$.

2. Using the additivity again we have that

   $P(\{1, 2\}) = P(\{1\}) + P(\{2\}) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$,

   $P(\{1, 3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$,

   $P(\{2, 3\}) = P(\{2\}) + P(\{3\}) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$.
II. 10 points A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than $x$ hours is $x/2$, for all $0 \leq x \leq 1$. Given that the student is still working after 0.75 hour, what is the conditional probability that the full hour is used?

**Hint:** Denote by $L_x$ the event that the student finishes the exam in less than $x$ hours, $0 \leq x \leq 1$, and by $F$ the event that the student uses the full hour.

**Solution:** $F$ is the event that the student uses the full hour, which means that $F$ is also the event that he is not finished in less than an hour. On the other side $L_x$ is the event that the student is finished in less than $x$ hours, hence

$$P(F) = P(L_1^c) = 1 - P(L_1) = 1 - 1/2 = 1/2.$$ 

Now, we are looking for the probability that the full hour is used, given that the student is still working after 0.75 hours, hence

$$P(F | L_{0.75}^c) = \frac{P(F \cap L_{0.75}^c)}{P(L_{0.75}^c)} = \frac{P(F)}{1 - P(L_{0.75})} = \frac{1}{1 - (0.75/2)} = \frac{1/2}{0.625} = 0.8.$$
III. **10 points** Four independent flips of a fair coin are made. Let $X$ denote the number of heads obtained. Find and plot the probability density (mass) function of the random variable $X - 2$.

**Solution:** Since the four flips are independent the random variable $X$ is a Binomial with parameters $4$ and $\frac{1}{2}$ ($X \sim B(4, 1/2)$), that is:

$$X : S \longrightarrow \{0, 1, 2, 3, 4\}$$

and

$$P(X = k) = C_4^k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{4-k}, \ k = 0, \ldots, 4.$$ 

Hence,

$$P(X - 2 = k - 2) = P(X = k) = C_4^k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{4-k}, \ k = 0, \ldots, 4.$$ 

$Y := X - 2$ is a discrete random variable with values in $\{-2, -1, 0, 1, 2\}$, with probability mass function

$$p_Y(-2) = \left(\frac{1}{2}\right)^4, \ p_Y(-1) = 4 \left(\frac{1}{2}\right)^4, \ p_Y(0) = 6 \left(\frac{1}{2}\right)^4, \ p_Y(1) = 4 \left(\frac{1}{2}\right)^4, \ p_Y(2) = \left(\frac{1}{2}\right)^4.$$
VI. **10 points** A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win $1.10; if they are different colors, then you win -$1.00 (that is, you loose $1.00.) Calculate

1. the expected value of the amount you win.
2. the variance of the amount you win.

**Solution:** Let us denote by $X$ the random variable that denotes the amount you win. On the other side, let us denote by $R_i$ the event that the $i$-th ball is red and by $B_i$ the event that the $i$-th ball is blue, where $i = 1, 2$. Hence

$$P(X = -1.00) = P(R_1 \cap B_2) + P(R_2 \cap B_1),$$

and

$$P(X = 1.10) = 1 - P(X = -1.00).$$

On the other side

$$P(R_1 \cap B_2) = P(R_1)P(B_2 | R_1) = \frac{5}{10} \cdot \frac{5}{9}$$

and

$$P(R_2 \cap B_1) = P(B_1)P(R_2 | B_1) = \frac{5}{10} \cdot \frac{5}{9}.$$

Hence,

$$P(X = -1.00) = \frac{5}{9}, \text{ and } P(X = 1.10) = \frac{4}{9}.$$

1. $E(X) = (1.10)P(X = 1.10) + (-1.00)P(X = -1.00)$

$$= (1.10)\frac{4}{9} - 1.00 \cdot \frac{5}{9} = \frac{-0.6}{9} \approx -0.067.$$

2. $Var(X) = (1.10)^2P(X = 1.10) + (-1.00)^2P(X = -1.00)$

$$= (1.10)^2 \frac{4}{9} + (-1.00)^2 \frac{5}{9} \approx 1.089.$$

5
V. 10 points You have two coins, a fair one with probability of heads 1/2 and an unfair one with probability of heads 1/3, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely it is that it is the fair one?

**Hint:** Denote by:

- $F$ = the selected coin is fair,
- $U$ = the selected coin is unfair,
- $H$ = the coin lands heads up.

**Solution:** Since the two coins are identical, this means that $P(F) = P(U) = \frac{1}{2}$. On the other side, we have the following given information

\[
P(H|F) = \frac{1}{2}, \text{ and } P(H|U) = \frac{1}{3}.
\]

We are looking for the following conditional probability $P(F|H)$ that is given by the following formula:

\[
P(F|H) = \frac{P(F \cap H)}{P(H)} = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/3)(1/2)} = \frac{3}{5}.
\]
VI. (Bonus 10 points) A communication system consists of $n$ components, each of which will, independently, function with probability $p$. The total system will be able to operate effectively if at least one-half of its components function.

1. For what values of $p$ is a 5-component system more likely to operate effectively than a 3-component system?

2. In general, when is a $(2k + 1)$-component system better than a $(2k - 1)$-component system?
Appendix: Let us recall that

1. $X$ is a Bernoulli random variable with parameter $p$ if $P(X = 0) = 1 - p$, $P(X = 1) = p$.

2. $X$ is a Binomial random variable with parameter $n, p$ if $P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$, and $k = 1, \ldots, n$.

3. $X$ is a Poisson random variable with parameter $\lambda$ if $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$, and $k = 1, 2, \ldots$. 