Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
I. (10 points) Suppose that the random variable $X$ is equal to the number of hits obtained by a certain baseball player in his next 3 at bats. If $P(X = 1) = 0.3$, $P(X = 2) = 0.2$ and $P(X = 0) = 3P(X = 3)$. Find $E(X)$.

**Solution:** Since the probabilities sum to 1, we have that

$$1 = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= P(X = 1) + P(X = 2) + 4P(X = 3)$$
$$= 0.5 + 4P(X = 3)$$

Hence,

$$P(X = 3) = 0.5/4 = 1/8 = 0.125.$$ and $P(X = 0) = 3P(X = 3) = 3/8 = 0.375$.

Now,

$$E(X) = 0 \times P(X = 0) + 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3)$$
$$= 1(0.3) + 2(0.2) + 3(0.125) = 1.075.$$
II. (10 points) If the distribution function of $X$ is given by

$$
F(b) = \begin{cases} 
0 & b < 0, \\
1/3 & 0 \leq b < 1, \\
2/3 & 1 \leq b < 2, \\
1 & b \geq 2.
\end{cases}
$$

Calculate the probability mass function of $X$.

**Solution:** Let us denote by $p_X$ the probability mass function of $X$, then for every $i = 0, 1, 2$

$$
p_X(i) = P(X = i) = F(i) - \lim_{b \to i^-} F(b).
$$

Hence,

$$
p_X(0) = F(0) - \lim_{b \to 0^-} F(b) = 1/3 - 0 = 1/3.
$$

$$
p_X(1) = F(1) - \lim_{b \to 1^-} F(b) = 2/3 - 1/3 = 1/3.
$$

$$
p_X(2) = F(2) - \lim_{b \to 2^-} F(b) = 1 - 2/3 = 1/3.
$$

Hence the mass function is given by $p_X(0) = p_X(1) = p_X(2) = 1/3$. 

III. (10 points) A random variable $X$ has probability density function of the form

$$f_X(x) = \begin{cases} 
    cx^2 & 0 \leq x \leq 1, \\
    0 & \text{otherwise.}
\end{cases}$$

1. Find the constant $c$.

2. Find $P(X \leq a)$ for $0 \leq a \leq 1$.

3. Calculate $E(X)$.

4. Find $SD(X)$.

Solution:

1. $f$ is a density if $f(x) \geq 0 \ \forall x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. Hence

$$1 = \int_{-\infty}^{\infty} f(x)dx = c \int_{0}^{1} x^2dx = cx^3\bigg|_{0}^{1} = c/3$$

Hence

$$c = 3.$$

2. Let $0 \leq a \leq 1$

$$P(X \leq a) = \int_{-\infty}^{a} f(x)dx = \int_{0}^{a} 3x^2dx = x^3\bigg|_{0}^{a} = a^3$$

3. $E(X) = \int_{-\infty}^{\infty} xf(x)dx = 3 \int_{0}^{1} x^3dx = (3/4)x^4\bigg|_{0}^{1} = 3/4$

4. $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = 3 \int_{0}^{1} x^4dx = (3/5)x^5\bigg|_{0}^{1} = 3/5.$
Hence

\[ Var(X) = E(X^2) - [E(X)]^2 = \left(\frac{3}{5}\right) - \left(\frac{3}{4}\right)^2 = \frac{3}{80}. \]

and

\[ SD(X) = \sqrt{Var(X)} = \sqrt{\frac{3}{80}}. \]
VI. (10 points) Let $X$ be uniformly distributed over $(0, 1)$. Find the density function of the random variable $Y = X^3$.

Solution: Let us denote (respectively) by $F_X$ and $F_Y$ the distribution functions of $X$ and $Y$ and by $f_X$ and $f_Y$ the densities functions of $X$ and $Y$.

$$f_X(x) = \begin{cases} 
1 & 0 \leq x \leq 1, \\
0 & \text{otherwise}. 
\end{cases}$$

And

$$F_X(x) = \begin{cases} 
0 & x < 0, \\
x & 0 \leq x \leq 1, \\
1 & x \geq 1.
\end{cases}$$

Let us compute the distribution function of $Y$ and let $0 \leq y \leq 1$

$$F_Y(y) := P(Y \leq y) = P(X^3 \leq y)$$
$$= P(X \leq y^{\frac{1}{3}})$$
$$= F_X(y^{\frac{1}{3}}) = y^{\frac{1}{3}}.$$

Hence for $0 \leq y \leq 1$

$$f_Y = \frac{d}{dy} (F_Y(y))$$
$$= \frac{d}{dy} y^{\frac{1}{3}} = \frac{1}{3} y^{-\frac{2}{3}}.$$

For $y < 0$

$$F_Y(y) := P(Y \leq y) = 0,$$

hence, $f_Y = 0$. For $y > 1$

$$F_Y(y) := P(Y \leq y) = 1,$$

hence, $f_Y = 0$.

$$f_Y(y) = \begin{cases} 
\frac{1}{3} y^{-\frac{2}{3}} & 0 \leq y \leq 1, \\
0 & \text{otherwise}. 
\end{cases}$$
V. (10 points) Let \( Z \) a standard normal random variable. Show that for all \( x > 0 \)

1. \( P(Z > x) = P(Z < -x) \).

2. \( P(|Z| < x) = 2P(Z < x) - 1 \).

Solution:

1. 

\[
P(Z > x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-y^2/2} \, dy
\]

\[
\equiv - \frac{1}{\sqrt{2\pi}} \int_{-x}^{-\infty} e^{-z^2/2} \, dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-z^2/2} \, dz
\]

\[
= P(Z < -x)
\]

2. 

\[
P(|Z| < x) = P(-x < Z < x) = P \left( Z \in (-\infty, x) \cap (-x, \infty) \right)
\]

\[
= 1 - P \left( Z \in (-\infty, x) \setminus (-x, \infty) \right)^c
\]

\[
= 1 - P \left( Z \in (-\infty, x)^c \cup (-x, \infty)^c \right)
\]

\[
= 1 - P \left( Z \in (x, \infty) \cup (-\infty, -x) \right)
\]

\[
= 1 - \overbrace{P(Z > x)}^{P(Z < x)} - \overbrace{P(Z < -x)}^{P(Z > x)}
\]

\[
= 2P(Z < x) - 1.
\]
VI. (Bonus 10 points) Show that

\[ E(Y) = \int_0^\infty P(Y > y)dy - \int_0^\infty P(Y < -y)dy. \]

**Hint:** Show that

\[
\int_0^\infty P(Y < -y)dy = -\int_{-\infty}^0 x f_Y(x)dx
\]

\[
\int_0^\infty P(Y > y)dy = \int_0^\infty x f_Y(x)dx.
\]