Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
A. (15pts) Let us define the following differential equation

\[ y'' + xy + y = 0 \]  

(1)

1. Seek power series solutions of the given differential equation about the point \( x_0 = 0 \): find the recurrence relation.

2. Find the first four terms in each of the solutions \( y_1 \) and \( y_2 \).

3. By evaluating the Wronskian \( W(y_1, y_2)(0) \), show that \( y_1 \) and \( y_2 \) form a fundamental set of solutions.

\[ y = \sum_{n \geq 0} a_n x^n \]

\[ y' = \sum_{n \geq 1} n a_n x^{n-1} \]

\[ y'' = \sum_{n \geq 2} n(n-1) a_n x^{n-2} \]

Substitute in (1), we get

\[ \sum_{n \geq 2} n(n-1) a_n x^{n-2} + \sum_{n \geq 1} n a_n x^{n-1} + \sum_{n \geq 0} a_n x^n = 0. \]

\[ \Rightarrow \text{using index shift for} \]

\[ \sum_{m \geq 0} (m+2)(m+1) a_{m+2} x^m + \sum_{n \geq 1} n a_n x^n + \sum_{n \geq 0} a_n x^n = 0. \]

\[ \Rightarrow \sum_{n \geq 0} (n+2)(n+1) a_{n+2} x^n + \sum_{n \geq 0} n a_n x^n + \sum_{n \geq 0} a_n x^n = 0. \]

\[ \Rightarrow \sum_{n \geq 0} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n\right] x^n = 0. \]

Hence \( (n+2)(n+1) a_{n+2} + (n+1) a_n = 0 \) \( n = 0, 1, 2, \ldots \)

\[ a_{n+2} = \frac{-a_n}{n+2} \]

\[ a_2 = \frac{-a_0}{2} \]

\[ a_3 = \frac{-a_1}{3} \]
\[
\begin{align*}
\text{For } n = 3: & \quad q_3 = -\frac{a_3}{3} = -\frac{1}{3} \left( -\frac{a_1}{3} \right) = \frac{a_1}{3.5}, \\
\text{For } n = 4: & \quad q_4 = -\frac{a_4}{4} = -\frac{a_0}{2.4.6}, \\
\text{For } n \geq 2: & \quad a_4 = -\frac{a_2}{4} + \frac{a_0}{2.4}, \\
\text{For } n \geq 3: & \quad a_7 = -\frac{a_5}{7} = -\frac{a_1}{3.5.7}, \\
\text{For } n \geq 6: & \quad a_7 = -\frac{a_6}{7} + \frac{a_0}{2.4.6.8}, \\
\text{In general} & \quad a_{2n} = \frac{(-1)^n a_0}{2.4.6...2n}, \\
& \quad a_{2n+1} = \frac{(-1)^n a_1}{1.3.5...(2n+1)}, \\
\end{align*}
\]

\[
\begin{align*}
y_3(x) &= \sum q_n x^n = a_0 + a_1 x + a_3 x^2 + a_5 x^3 + a_7 x^4 + a_9 x^5 + \cdots, \\
&= a_0, \\
y_1(x) &= 1 - \frac{1}{2} x^2 + \frac{1}{2.4} x^4 - \frac{1}{2.4.6} x^6 + \cdots, \\
y_2(x) &= x - \frac{1}{3} x^3 + \frac{1}{3.5} x^5 - \frac{1}{3.5.7} x^7 + \cdots, \\
3) \quad \text{W}(y_1, y_2) &= \begin{vmatrix} y_1^{(0)} & y_2^{(0)} \\ y_1^{(1)} & y_2^{(1)} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0, \\
&= \{y_1, y_2\} \text{ if fundamental set.}
\end{align*}
\]
B. (10pts) Let us define the Euler equation

$$2x^2y'' + 3xy' - y = 0. \quad (\text{A})$$

1. Solve the given differential equation for $x > 0$

2. Study the qualitative behavior of the solution near the singular point $x = 0$.

**Sol:** 1) Similar to an Euler Eq. Assume that we have a Nd of the form $y = x^r$. Then, $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Hence, if we substitute in (A) then,

$$x^r(2r(r-1) + 3r - 1) = x^r(2r^2 + r - 1) = x^r(2r-1)(r+1) = 0.$$ 
Hence, $r_1 = \frac{1}{2}, r_2 = -1$, and the general solution is given by

$$y = c_1 x^{r_1} + c_2 x^{r_2} = c_1 x^{1/2} + c_2 x^{-1} \quad x > 0$$

where $c_1, c_2$ are arbitrary.

2) Near $x = 0$,

$$y(x) = c_1 x^{1/2} + c_2 x^{-1}.$$ 

$$\lim_{x \to 0^+} y(x) = \lim_{x \to 0^+} c_1 x^{1/2} + c_2 x^{-1} = \infty \quad (\text{depending on } c_2)$$

Hence.
C. (15pts) The Chebyshev equation is

\[(1 - x^2)y'' - xy' + \alpha^2 y = 0,\]

where \(\alpha\) is a constant.

1. Show that \(x = 1\) and \(x = -1\) are regular singular points.

2. Find the exponents at each of the singularities \(x = 1\) and \(x = -1\).

3. Find two solutions about \(x = 1\).

\[\text{Sol: } (1) \quad \alpha = 1, \quad P(x) = 1 - x^2, \quad Q(x) = -x, \quad R(x) = x^2.\]

\[\lim_{x \to 1} \frac{Q(x)}{P(x)} = \lim_{x \to 1} \frac{(x-1)(-x)}{(1-x^2)} = \lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2} = \rho.\]

\[\lim_{x \to 1} \frac{(x-1)^2 R(x)}{P(x)} = \lim_{x \to 1} \frac{(x-1)^2 x^2}{(1-x^2)} = \lim_{x \to 1} \frac{(x-1)x^2}{x+1} = 0 = \rho.\]

Hence, \(x = 1\) is a regular singular point.

\[\alpha = -1\]

\[\lim_{x \to -1} \frac{Q(x)}{P(x)} = \lim_{x \to -1} \frac{(x+1)(-x)}{(1-x^2)} = \lim_{x \to -1} \frac{-x}{1-x} = \frac{1}{2} = \rho.\]

\[\lim_{x \to -1} \frac{(x+1)^2 R(x)}{P(x)} = \lim_{x \to -1} \frac{(x+1)^2 x^2}{1-x^2} = \lim_{x \to -1} \frac{x^2(x+1)}{1-x} = 0 = \rho.\]

Hence, \(x = -1\) is a regular singular point.

(2) The indicial equation for \(\alpha = 1\) is given by

\[F(r) = r(r-1) + \rho_0 r + g_0 = r(r-1) + \frac{1}{2} r = 0.\]

\[\Rightarrow r^2 - r + \frac{1}{2} r = 0 \Rightarrow r^2 - \frac{1}{2} r = 0 \Rightarrow r \left( r - \frac{1}{2} \right) = 0 \Rightarrow r = 0, \quad r = \frac{1}{2}.\]
Hence, the exponents of the singularity are $r = 0, r = \frac{1}{2}$.

For $n = 1$

\[ F(s) = r(r-1) + p_0 r + q_0. \]

\[ = r(r-1) + \frac{1}{2} r = 0 \implies r = 0 \text{ or } r = \frac{1}{2}. \]

The exponents of the singularity for $n = 1$ are $r = 0$ or $r = \frac{1}{2}$.

3) \[ y_1(x) = \frac{1}{n-1} \left[ 1 + \sum_{n=0}^{\infty} \frac{(-1)^n (1+2x) \cdots (2n-1+2x)(1-2x) \cdots (2n-1-2x)}{(x-1)^n} \right]. \]

\[ y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1+2x) \cdots (2n-1+2x)(1-2x) \cdots (2n-1-2x)}{(2n+1)!} \left[ (n-1)^n \right]. \]