Sets, Relations & Functions

Fill-in the blank:
for all \( u \in U \), we have that \( u \in V \).
\( u \in U \)
\( u \in V \)
\( U \subseteq V \)
\( V \subseteq U \)
\( u \in V \)
\( v \in U \)
\( U \cap V \)
\( \{ x : x \in U \text{ and } x \in V \} \)
\( U \cup V \)
\( \{ x : x \in U \text{ or } x \in V \} \)
\( \{(u, v) : u \in U \text{ and } v \in V \} \)
\( \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\} \)
\( \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\} \)

\( S \times S \)
\( aRa \) for all \( a \in S \)
\( aRb \) implies \( bRa \)
\( aRb \) and \( bRc \) implies \( aRc \)
It is not symmetric \( 3 \leq 5 \) but \( 5 \) is not \( \leq 3 \)
It is reflexive, symmetric and transitive

\( \{ x : xRs \} \).
\( [2] = \{ \ldots, -3, 2, 7, 12, \ldots \} \)
\( [2] = [7] \)
\( [s_1] \cap [s_2] = \emptyset \).
Suppose that \([s_1] \cap [s_2] \neq \emptyset\) and let \( x \) be in the intersection. Thus \( xRs_1 \) and \( xRs_2 \). Suppose that \( y \in [s_1] \). So \( yRs_1 \). Since \( R \) is symmetric, \( s_1Rx \). Since \( R \) is transitive and \( yRs_1, s_1Rx \), and \( xRs_2 \) we have that \( yRs_2 \). Thus, \( y \in [s_2] \) and \([s_1] \subseteq [s_2] \). Similarly, \([s_2] \subseteq [s_1] \). So \([s_1] = [s_2] \).

\( [0], [1], [2], [3], [4] \)

subset of \( X \times Y \)
for each \( x \in X \) there is exactly one \( y \) with \((x, y)\) in the subset
domain
co-domain
for each \( y \) there is at most one \( x \) with \( f(x) = y \)
for each \( y \) there is at least one \( x \) with \( f(x) = y \).
if \( f \) is both one to one and onto
one-to-one correspondence
the map \( f(x) = 2x \) is a 1-1 correspondence between the integers and the even integers
see notes
see notes

Section 0

29. \( R \) is not reflexive, since \( 0 \cdot 0 \) is not greater than 0
   \( R \) is symmetric: Assume \( mRn \). Then \( mn > 0 \). Since \( nm = mn, \) \( nm > 0 \).
   So \( nRm \).
   \( R \) is transitive: Assume \( mRn \) and \( nRp \). Then \( mn > 0 \) and \( np > 0 \). This
   means that both \( m \) and \( n \) are positive or both are negative, and similarly
   for \( n \) and \( p \). Thus, either all of \( m, n, p \) are positive, or all are negative. In
   either case \( mp > 0 \). Thus, \( mRp \).

32. To see that \( R \) is reflexive, let \( x \in \mathbb{R} \). Then \( |x - x| = |0| = 0 \leq 3 \). So \( xRx \).
   To see that \( R \) is symmetric, assume that \( xRy \). Then \( |x - y| \leq 3 \). Since
   \( |y - x| = |x - y| \), we see that \( |y - x| \leq 3 \). So \( yRx \).
   \( R \) is not symmetric because \( 0R3 \), and \( 3R6 \), but \( 0 \) is not related to \( 6 \).

36a. To see that \( R \) is reflexive let \( x \) be an integer. Then \( x - x = 0 = 0 \cdot n \).
   So \( xRx \).
   To see that \( R \) is symmetric, assume that \( xRy \). Then \( x - y = nq \) for some
   integer \( q \). Multiplying by \(-1 \) we see that \( y - x = n(-q) \). So, \( yRx \).
   To see that \( R \) is transitive, assume that \( xRy \) and \( yRz \). Then \( x - y = qq' \)
   for some integer \( q \) and \( z - y = nq' \) for some integer \( q' \). Adding these
   equations together gives: \( x - z = n(q - q') \). Thus, \( xRz \).

Binary operations & structures
Fill-in the blank
\( S \times S \)
\s_1 \oplus s_2 = s_2 \oplus s_1 \) for all \( s_1, s_2 \in S \).
\( s_1 \oplus (s_2 \oplus s_3) = (s_1 \oplus s_2) \oplus s_3 \) for all \( s_1, s_2, s_3 \in S \)
\( e \oplus s = s \oplus e \) for all \( s \in S \)
\( t \in S \)
\( s \oplus t = e = t \oplus s \).
Suppose that \( t \) and \( t' \) are inverses of \( S \). Consider \( t \oplus (s \oplus t') \). Since \( t' \) is an
inverse of \( s, t \oplus (s \oplus t') = t \oplus e = t \). Since \( \oplus \) is associative and \( t \) is an inverse of
\( s, t \oplus (s \oplus t') = (t \oplus s) \oplus t' = e \oplus t' = t'. \) So \( t = t' \).

Section 1

19. To solve \( z^3 = -27i \), we first find the polar representation of \( -27i \). \( -27i \)
points at \( 3\pi/2 \) and has length \( 27 \). So \( -27i = 27e^{3\pi i/2} \). Write \( z = re^{i\theta} \).
   Then \( z^3 = r^3e^{3i\theta} \). So \( r^3 = 27 \) and \( r > 0 \). Thus, \( r = 3 \). Also, \( 3\theta = 3\pi/2 + \)
2πk for some integer k. The distinct (mod 2π) possibilities for θ are π/2, π/2 + 2π/3 = 7π/6 and π/2 + 4π/3 = 11π/6. Thus, z = 3e^π/2 = 3i, z = 3e^{7π/6} = 3(−√32−i/2) = −3√2/2−3i/2, or z = 3e^{11π/6} = 3√3/2−3i/2.

35. Since ψ goes to 5, and ψ^2 must go to 5 + 5 = 2, and ψ^3 must go to 5 + 5 + 5 = 7. Similarly, ψ^4 goes to 4, ψ^5 goes to 1, ψ^6 goes to 6, ψ^7 goes to 3, and 1 = ψ^8 goes to 0.

Section 2

3. (b * d) * c = e * c = a, b * (d * c) = b * b = c. Yes, * is not associative.

5. We just complete the table to make it symmetric. So we take a * d = d, b * c = a, d * b = c and d * c = b.

7. Not commutative: 3 * 5 = 3 − 5 which isn’t 5 * 3 = 5 − 3.

Not associative: 1 * (2 * 3) = 1 − (2 − 3) − 1 − (−1) = 2, and (1 * 2) * 3 = −1 − 3 = −4

9. Is commutative: a * b = ab/2 = ba/2 = b * a

Is associative: a * (b * c) = a*(bc/2) = a(bc/2)/2 = abc/4, and (a * b) * c = (ab/2) * c = (abc/2)/2 = abc/4.

11. Not commutative 2 * 3 ≠ 3 * 2 Not associative 1 * (2 * 3) = 1^9 = 1, and (1 * 2) * 3 = 2^3 = 8.

24. (a) False. We could have S = Z and * being normal multiplication.

(b) True. Set x = b * c. Then a * x = x * a. Now substitute in the value of x to get the desired result.

(c) False. We could take S be be the 2 by 2 matrices, and * to be normal multiplication. If the statement were true, then by taking c to be the identity matrix, we would have that ab = ba for all 2 by 2 matrices—which is certainly false.

(d) False. We can define important binary operations on sets, on functions, on geometric objects,

(e) False. It is commutative if a * b = b * a for all a and b

(f) True. If the only element is Bob, then Bob * Bob must equal Bob. So it’s table is is symmetric, and for associativity the only thing we must check is Bob * (Bob * Bob) = (Bob * Bob) * Bob? which is true since both computations yield Bob.

Isomorphisms

Fill-in the blank:
f(s_1) ⊕ f(s_2) for all s_1, s_2 ∈ S.

ln(x)
1 → 0, i → 1, −1 → 2, −i → 3.
First assume that \((S, \ast)\) is commutative. Take \(t_1, t_2 \in T\). Since \(f\) is onto, there exist \(s_1, s_2 \in S\) with \(f(s_1) = t_1\) and \(f(s_2) = t_2\). Since \(\ast\) is commutative, \(s_1 \ast s_2 = s_2 \ast s_1\). Applying \(f\) gives \(f(s_1 \ast s_2) = f(s_2 \ast s_1)\). Since \(f\) preserves operations, \(f(s_1) \ast f(s_2) = f(s_2) \ast f(s_1)\). So \(t_1 \ast t_2 = t_2 \ast t_1\), and \(\ast\) is commutative.

Next assume that \((T, \oplus)\) is commutative. Take \(s_1, s_2 \in S\). Let \(t_1 = f(s_1)\) and \(t_2 = f(s_2)\). Since \(\oplus\) is commutative, \(t_1 \oplus t_2 = t_2 \oplus t_1\). Thus \(f(s_1) \oplus f(s_2) = f(s_2) \oplus f(s_1)\). Since \(f\) preserves operations, \(f(s_1 \ast s_2) = f(s_2 \ast s_1)\). Since \(f\) is one to one, \(s_1 \ast s_2 = s_2 \ast s_1\). So \(\ast\) is commutative.

Section 3

1. One must check that \(\phi\) is one to one, \(\phi\) is onto, and \(\phi(x \ast y) = \phi(x) \ast' \phi(y)\) for all \(x, y \in S\).

3. \(\phi\) is one to one: Suppose that \(\phi(x) = \phi(y)\). Then \(2x = 2y\), so \(x = y\).

\(\phi\) preserves operations: Let \(x, y \in \mathbb{Z}\). Then \(\phi(x + y) = 2(x + y) = 2x + 2y = \phi(x) + \phi(y)\).

\(\phi\) is not onto, since there is not \(x \in \mathbb{Z}\) such that \(f(x) = 2\). So \(f\) is not an isomorphism.

5. \(\phi\) is 1-1: Assume \(\phi(x) = \phi(y)\). Then \(x/2 = y/2\). Multiplying by 2 gives \(x = y\).

\(\phi\) is onto: Let \(x \in \mathbb{Q}\). Then \(2x \in \mathbb{Q}\) and \(\phi(2x) = (2x)/2 = x\).

\(\phi\) preserves operations: Let \(x, y \in \mathbb{Q}\). Then \(\phi(x + y) = (x + y)/2 = x/2 + y/2 = \phi(x) + \phi(y)\).

Thus, \(\phi\) is an isomorphism.

7. \(\phi\) is 1-1 and onto, but \(\phi\) does not preserve operations because \(\phi(1+1) = 2^3\), but \(\phi(1) + \phi(1) = 1^3 + 1^3 = 2\).

29. See the argument above showing that if \((S, \ast)\) and \((T, \oplus)\) are isomorphic groups, then \(S\) is commutative if and only if \(T\) is commutative.

31. Let \((S, \ast)\) and \((T, \oplus)\) be isomorphic binary structures with isomorphism \(f\).

First suppose that for each \(c \in S\) the equation \(x \ast x = c\) has a solution in \(S\). We must show that for each \(d \in T\) the equation \(y \oplus y = d\) has a solution in \(T\). Take \(d \in T\). Since \(f\) is onto, there is a \(c \in S\) with \(f(c) = d\). By assumption there is an \(x \in S\) with \(x \ast x = c\). Applying \(f\), we get \(f(x \ast x) = f(c)\). Since \(f\) preserves operations, we get \(f(x) \oplus f(x) = f(c)\). So \(y = f(x)\) is a solution to \(y \oplus y = d\).

Now suppose that for each \(d \in S\) the equation \(y \oplus y = d\) has a solution in \(T\). We must show that for each \(c \in S\) the equation \(x \ast x = c\) has a solution in \(S\). Let \(d = f(c)\). Then there exists a \(y \in T\) such that \(y \oplus y = d\). Since \(f\) is onto, there is an \(x \in S\) with \(f(x) = y\). So \(f(x) \oplus f(x) = d\). Since \(f\)
preserves operations, \( f(x \ast x) = d = f(c) \). Since \( f \) is one to one, \( x \ast x = c \), as desired.

**Groups**

**Fill-in the blank**

an associative binary operation on \( G \)

\[
x \ast y = e = y \ast x
\]

Suppose both \( e \) and \( f \) are identities of \( G \). Since \( e \) is an identity \( e \ast f = f \). Since \( f \) is an identity, \( e \ast f = e \). So \( e = f \).

**Identity**

Suppose that \( y \) and \( z \) are inverses of \( x \). Consider \( y \ast (x \ast z) \). On the one hand, since \( z \) is an inverse of \( x \), \( y \ast (x \ast z) = y \ast e = y \). On the other hand, since \( y \) is an inverse of \( x \), and \( \ast \) is associative: \( y \ast (x \ast z) = (y \ast x) \ast z = e \ast z = z \). So \( y = z \).

\( x^{-1} \) for general groups, or \(-x\) for abelian groups

\[
a = b
\]

Suppose that \( a \ast x = b \ast x \). Since we are in a group, \( x \) has an inverse, say \( y \). Post-multiplying by \( y \) gives: \( (a \ast x) \ast y = (b \ast x) \ast y \). By associativity: \( a \ast (x \ast y) = b \ast (x \ast y) \). Since \( x \ast y = e \) we have \( a = a \ast e = b \ast e = b \). So \( a = b \).

\( \bar{0}, \bar{1}, \ldots, \bar{n} - 1 \).

\( i + j \) is \( i + j < n \), and \( i + j - n \) otherwise.

The subsets of \( X \).

\[
\begin{bmatrix}
\text{Even} & \text{Odd} \\
\text{Even} & \text{Odd} \\
\text{Odd} & \text{Odd} & \text{Even}
\end{bmatrix}
\]

\[
\begin{array}{c|cccc}
\text{e} & \text{u} & \text{v} & \text{w} \\
\hline
\text{e} & \text{e} & \text{u} & \text{v} & \text{w} \\
\text{u} & \text{u} & \text{e} & \text{w} & \text{v} \\
\text{v} & \text{v} & \text{w} & \text{e} & \text{u} \\
\text{w} & \text{w} & \text{v} & \text{u} & \text{e}
\end{array}
\]

its operation is commutative

\((\mathbb{Z}, +)\) is abelian

the rigid motions of the triangle

**Section 4**

1. This is not a group. Multiplication is associative, 1 is an identity, but 2 does not have an multiplicative inverse in \( \mathbb{Z} \).

3. This is not a group. 1 is an identity, \( 1/a \) is the inverse of \( a \); not associative because \( (1 \ast (2 \ast 2)) = 1 \ast 2 = \sqrt{2} \), but \( (1 \ast 2) \ast 2 = \sqrt{2} \ast 2 = \sqrt{\sqrt{2} \ast 2} \).
5. Not a group: There is not identity; Since there is no identity we can’t talk about inverse; not associative \(1 \ast (2 \ast 3) = 1/(2/3) = 3/2\), and \((1 \ast 2) \ast 3 = (1/2)/3 = 1/6\).

7. \((\mathbb{Z}_{1000}, +)\), or \((U_{1000}, \cdot)\)

10. (a) \(m_1n + (m_2n + m_3n) = (m_1n + m_2n) + m_3n\) because usual addition is associative. 0 = 0n is a identity. The inverse of \(mn\) is \(-mn = (-m)n\) is in the set.

(b) Consider the map \(g : \mathbb{Z} \to n\mathbb{Z}\) by \(g(x) = nx\). \(g\) is one-to-one: Assume \(g(x) = g(y)\). Then \(nx = ny\). Since \(n \geq 0\), we can divide \(n\) to get \(x = y\).

\(g\) is onto: Let \(mn \in n\mathbb{Z}\). Then \(g(m) = mn\).

\(g\) preserves operations: Let \(x, y \in \mathbb{Z}\). Then \(g(x+y) = n(x+y) = nx+ny = g(x) + g(y)\).

25. (a). False.

(b). True. One element \(e\) has to be an identity. Now the only table that satisfies the cancellation laws is

\[
\begin{array}{ccc}
& e & x & y \\
e & e & x & y \\
x & x & y & e \\
y & y & e & x \\
\end{array}
\]

(c). True. \(a \ast x = b\) has the solution \(x = a^{-1} \ast b\).

(g). True. See part (b)

(h). True. If \(a \ast x \ast b = c\) and \(a \ast y \ast b = c\), then \(a \ast x \ast b = a \ast y \ast b\). By left cancellation, \(x \ast b = y \ast b\). By right cancellation, \(x = y\).

(i). False. It doesn’t have an identity

(j). True.

Subgroups

Fill-in the blank

\(H\) is a group itself with the operation of \(G\)

\{\(R_0\), \(\{R_0, R_{120}, R_{240}\}\), \(\{R_0, F_A\}, \{R_0, F_B\}\), \(\{R_0, F_C\}\) and \(G\)

\(n\mathbb{Z}\) \(n\) a nonnegative integer

\(e \in H\)

the operation

inverses

Since \(H\) is a subgroup \(e \in H\). Similarly \(e \in K\). So \(e \in H \cap K\). Let \(u, v \in H \cap K\).

Since \(H\) is a subgroups \(u \ast v \in H\). Similarly, \(u \ast v \in K\). Thus, \(u \ast v \in H \cap K\).

Also \(u^{-1} \in H\), and \(u^{-1} \in K\). So \(u^{-1} \in H \cap K\).

\(e = g^0\). \(g^i \ast g^j = g^{i+j}\), and \(g^i\)'s inverse is \(g^{-i}\).
Section 5

1. Yes. 0 is real, the sum of two real numbers is real; the inverse of a real number is real.

5. Yes 0 is a rational multiple of \( \pi \), \( a\pi + b\pi = (a + b)\pi \) and \( a + b \) is rational when \( a \) and \( b \) are rational; \(-a\pi\) is the inverse of \( a\pi \).

6. No. \( \pi^2 + \pi^3 \) is not equal to \( \pi^n \) for any integer \( n \).

27. \( \bar{3} \), \( \bar{3} + \bar{3} = \bar{2} \), \( \bar{2} + \bar{3} = \bar{1} \), \( \bar{1} + \bar{3} = \bar{0} \). So order is 4.

36. 

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</table>

(b) \( \langle 0 \rangle = \{0\} \), \( \langle 1 \rangle = \langle 5 \rangle = \mathbb{Z}_6 \), \( \langle 2 \rangle = \langle 4 \rangle = \{0, 2, 4\} \), \( \langle 3 \rangle = \{0, 3\} \).

(c) 1 and 6

(d) Omit

(b). False.
(c). True
(d). False. (The group with just one element just has one improper subgroup)
(e). False. 2 doesn’t generate \( \mathbb{Z}_4 \)
(f). False. \( \mathbb{Z}_3 \) has two generators
(g). False. Consider \( \mathbb{Z} \)
(h). False. The odds is a subset of the integers, but is not a subgroup.
(i). True. it is generated by \( \bar{1} \)
(j). False. See answer to h.

51. First \( e \in H_a \): \( ae = a = ea \). So \( ae = ea \), and \( e \in H_a \).
Second \( H_a \) is closed under multiplication. Suppose that \( x, y \in H_a \). Then \( xa = ax \) and \( ya = ay \). Now \( (xy)a = x(ya) = x(ay) = (xa)y = (ax)y = a(xy) \). So \( xy \in H \).
Finally, suppose that \( x \in H_a \). Then \( ax = xa \). Pre-multiply by \( x^{-1} \) and post-multiply by \( x^{-1} \) to get \( x^{-1}a = ax^{-1} \). So \( x^{-1} \in H_a \).

54. See answer in the fill-in-the-blank section.