Math 2310

Exam I

DEPARTMENT OF MATHEMATICS
University of Wyoming
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Name (Print): key
Instructor: Long Lee

Section: 03

Directions:

- Do not open this exam until you are told to begin.

- This exam consists of 5 questions on 5 pages (excluding this cover sheet and the last page). The last page is blank and is for your use as a worksheet.

- This is a closed-book, closed-note exam. You are allowed a single 3" × 5" notecard (both sides). Calculators and all other aids are not allowed.

- Do not separate any of the exam pages.

- Please read the instructions for each exercise carefully. Show an appropriate amount of work for each exercise so that the grader can see not only the answer but also how you obtained the answer. Make sure that any graphs or tables you use in your answers are clearly labeled and accompanied by appropriate explanations. Explanations should be written in grammatically correct, complete English sentences.

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1. Find the solution to the initial value problem:

\[ \frac{dy}{dx} = \frac{2x}{1 + 3y^2}, \quad y(0) = 1. \]

\[(1 + 3y^2) \, dy = 2x \, dx\]

\[y + y^3 = x^2 + C\]

Applying the I.C., we have \( C = 2 \).

\[ y + y^3 = x^2 + 2 \]
2. Consider the initial value problem:

\[2y' - y = e^{t/3}, \quad y(0) = a.\]

(a) Solve the initial value problem.

(b) There are two different long-time \((t \to \infty)\) behaviors of the solution found in (a). Which of these two behaviors occurs depends on the choice of the initial value \(a\). In fact, there is a critical value \(a_0\) such that the solution \(y\) satisfies

\[
\lim_{t \to \infty} y(t) = \begin{cases} 
\infty & \text{if } a > a_0, \\
-\infty & \text{if } a < a_0.
\end{cases}
\]

Find the critical value \(a_0\) exactly.

\[
y' - \frac{1}{2} y = \frac{1}{2} e^{t/3}
\]

The integrating factor is

\[\mu(t) = e^{\int \frac{1}{2} dt} = e^{-\frac{t}{2}}\]

Hence,

\[
\frac{d}{dt} \left( e^{-\frac{t}{2}} y \right) = \frac{1}{2} e^{t/3} \cdot e^{-\frac{t}{2}} = \frac{1}{2} e^{-t/6}
\]

Integrating on both sides

\[e^{-\frac{t}{2}} y = \int \frac{1}{2} e^{-t/2} dt = -\frac{6}{2} e^{-t/6} + C = 3e^{-t/6} + C
\]

\[y = -3e^{\frac{t}{3}} + Ce^{t/2}\]

Applying the initial condition

\[y(0) = a\] we have

\[-3 + C = a\]

\[\therefore C = 3 + a\]
Hence

\[ y(t) = -3 e^{\frac{t}{3}} + (a+3) e^{\frac{t}{2}} \]

If \( a+3 > 0 \), or \( a > -3 \)

\[ \lim_{{t \to \infty}} y(t) = +\infty \]

Otherwise, \( a \leq -3 \)

\[ \lim_{{t \to \infty}} y(t) = -\infty \]

Hence the critical value of \( a \) \( a_c = -3 \).
3. Consider the autonomous differential equation, \( y' = f(y) \):

\[
\frac{dy}{dt} = -2(1 - \frac{y}{4})y.
\]

(a) Determine the critical (equilibrium) points.
(b) Sketch \( f(y) \) versus \( y \) and classify each critical point as asymptotically stable, unstable, or semistable?
(c) Without solving the equation, draw the phase line and sketch several graphs of solutions in the \( ty \)-plane.

(a) The equilibrium points are roots of \( f(y) = 0 \)

where, \( f(y) = -2(1 - \frac{y}{4})y = -2y + \frac{y^2}{2} \)

Hence \( (y, f(y)) = (0, 0) \) & \( (4, 0) \) are the critical points

(b) \( f'(y) = 0 \Rightarrow -2 + y = 0, \ y = 2 \). The local minimum is at \( (2, -2) \)

\( y = 0 \) is asymptotically stable, \( y = 4 \) is unstable.
(c) \[ y'' = f'(y) \cdot f(y) \]

1. \( y'' = 0 \) at \( y = 2 \) for which \( f(2) = 0 \)
   Hence \( y = 2 \) is an inflection point.

2. \( y'' > 0 \) for \( 0 < y < 2 \), since \( f'(y) < 0, f(y) > 0 \)
   (concave up) when \( 0 < y < 2 \)

Similarly,

3. \( y'' < 0 \) for \( 2 < y < 4 \), (concave down)

Note 4. \( y'' < 0 \), if \( y < 0 \) (concave down)

5. \( y'' > 0 \) if \( y > 4 \) (concave up)
4. The equation \((x^2 + y) \, dx + f(x) \, dy = 0\) is known to have an integrating factor \(\mu(x) = x\). Find all possibilities for \(f\).

For exact equations, we require

\[
(\mu M)_y = (\mu N)_x
\]

Since

\[
\mu(x) = x
\]

\[
M = x^2 + y
\]

\[
N = f(x)
\]

we have

\[
\left[ x(x^2 + y) \right]_y = \left[ x f(x) \right]_x
\]

or

\[
\frac{d}{dx} [x f(x)] = x
\]

Integrating on both sides

\[
x f(x) = \frac{1}{2} x^2 + C
\]

\[
f(x) = \frac{1}{2} x + \frac{C}{x}, \text{ for } x \neq 0, \text{ and arbitrary } C.
5. Find the solution to the initial value problem:

\[ y' = |x|, \quad y(-1) = 2. \]

[Hint: To obtain the solution of this problem, you will need to discuss the domain of the solution that contains the initial condition.]

There will be two functions joining at \( x=0 \). For the solution, we first note that:

\[
\int_2^y y'dy = \int_1^x |x| \, dx
\]

1) When \( x<0 \), we have

\[
\int_2^y y'dy = \int_1^x (-x) \, dx
\]

\[
y \bigg|_2^y = -\frac{1}{2}x^2 \bigg|_1^x
\]

Hence \( y = -\frac{1}{2}x^2 + \frac{5}{2} \)

2) When \( x>0 \), we have

\[
\int_2^y y'dy = \int_1^x |x| \, dx
\]

\[
= \int_1^0 (-x) \, dx + \int_0^x (x) \, dx
\]

\[
y = -\frac{1}{2}x^2 \bigg|_1^0 + \frac{1}{2}x^2 \bigg|_0^x
\]

\[
y = \frac{1}{2}x^2 + \frac{5}{2}
\]
Hence the solution for the differential equation is

\[ y(x) = \begin{cases} 
-\frac{1}{2}x^2 + \frac{5}{2} & x \leq 0 \\
\frac{1}{2}x^2 + \frac{5}{2} & x > 0 
\end{cases} \]  

(This satisfies the initial condition.)
This page is left blank for a worksheet. Work on this page will NOT be graded unless you explicitly request so ON THE PAGE WHERE THE PROBLEM IS STATED.