Solutions to HW1

1. Denote by $S_3$ the nonabelian group of order 6. How many homomorphisms are there from $S_3$ to $S_3$? List them all.

   **Solution.** Let $\varphi : S_3 \to S_3$ be a homomorphism. Then $\ker \varphi$ is one of the three normal subgroups of $S_3$. We consider these three cases in turn.

   (i) $\ker \varphi = S_3$. In this case $\varphi$ is trivial, i.e. $\varphi(\sigma) = 1$ for all $\sigma \in S_3$.

   (ii) $\ker \varphi = \langle (123) \rangle$. In this case the image of $\varphi$ is a subgroup of order 2, of which there are three choices. One choice gives

   $$\varphi(\sigma) = \begin{cases} (1), & \text{if } \sigma \in \langle (123) \rangle, \text{ i.e. } \sigma \text{ is even;} \\ (12), & \text{if } \sigma \notin \langle (123) \rangle, \text{ i.e. } \sigma \text{ is odd.} \end{cases}$$

   Two other choices arise by replacing $(12)$ by $(13)$ or $(23)$ in the latter definition.

   (iii) $\ker \varphi = 1$. In this case $\varphi$ is an automorphism of $S_3$. There are six inner automorphisms, all of which are distinct. In fact, these are all the automorphisms of $S_3$. To prove this, we will prove that $S_3$ has at most six automorphisms. Note that every automorphism $\varphi$ permutes the set $T = \{ (12), (13), (23) \}$ of elements of order 2. There are exactly $3! = 6$ permutations of $T$, each of which extends to at most one automorphism of $S_3$ (since $\langle T \rangle = S_3$) as required.

   Combining these cases, we see that there are exactly ten homomorphisms $S_3 \to S_3$.

2. Let $G$ be a group, and let $A, B$ be normal subgroups of $G$ such that $G = AB$. Let $N = A \cap B$. Show that $G/N \cong A/N \times B/N$.

   **Solution.** Note that $A/N = A/(A \cap B) \cong AB/B = G/B$; similarly $B/N \cong G/A$. It suffices to show that

   $$G/N \cong G/A \times G/B.$$ 

   Define $\varphi : G \to G/A \times G/B$ by $\varphi(g) = (gA, gB)$. If $g, g' \in G$ then

   $$\varphi(g)\varphi(g') = (gA, gB)(g'A, g'B) = (gg'A, gg'B) = \varphi(gg'),$$

   so $\varphi$ is a homomorphism. To show that $\varphi$ is surjective, consider a typical element $(g_1A, g_2B) \in G/A \times G/B$. By hypothesis, we can write $g_i = a_ib_i$ where $a_i \in A$, $b_i \in B$ for $i = 1, 2$. Now

   $$\varphi(a_2b_1) = (a_2b_1A, a_2b_1B) = (b_1A, a_2B) = (g_1A, g_2B).$$
Thus $\varphi$ is surjective as claimed. If $g \in \ker \varphi$ then
\[
\varphi(g) = (gA, gB) = (A, B)
\]
so that $g \in A \cap B = N$. Thus $\ker \varphi \subseteq N$, and clearly the reverse inclusion holds. By the First Isomorphism Theorem, we obtain
\[
G/N \cong G/A \times G/B \cong A/N \times B/N.
\]

3. Let $G$ be a group generated by two distinct elements $x, y$ of order 2.

(a) Show that $G$ has an abelian normal subgroup $H$ of index 2.

Solution. Since $x^2 = y^2 = 1$, we have
\[
G = \{1, x, y, xy, yx, yxy, yxyx, yxyyx, yxyxy, \ldots\}
\]
where some of the elements in this infinite list may possibly coincide. Let $h = xy$, so that $h^{-1} = yx$; then
\[
G = \{\ldots, h^{-2}, h^{-1}, 1, h, h^2, \ldots\} \cup \{\ldots, h^{-2}x, h^{-1}x, x, hx, h^2x, \ldots\} = H \cup Hx.
\]

If $H = G$ then $G$ is itself cyclic and so $G$ has a unique element of order 2; but then $x = y$, contrary to hypothesis. Hence $[G : H] = 2$ and $H \trianglelefteq G$.

(b) What are the possibilities for the order $|G|$? In each case indicate also the order $|Z(G)|$ of the centre. Justify your answers.

Solution. Let $h = xy$ as in (a). If $|h| = \infty$ then we have $|G| = \infty$. Otherwise we have $|G| = 2n$ where $n = |h| \geq 2$. Note that $h^x = xhx = yx = h^{-1}$ and similarly $h^y = h^{-1}$. Taking $k$-th powers gives
\[
(h^k)^x = h^{-k}; \quad (h^k)^y = h^{-k}
\]
so if $h^k$ commutes with both $x$ and $y$, then $h^{2k} = 1$, so $n$ divides $2k$. Similarly
\[
(h^kx)^x = h^{-k}x; \quad (h^kx)^y = h^{-k}x^y = h^{-k}yxy = h^{-k-2}x
\]
so if $h^kx$ commutes with both $x$ and $y$, then we obtain $h^{2k} = h^{2k+2} = 1$ and so $n = 2$. Summarizing, we obtain:

- If $|h| = 2$ (i.e. $|G| = 4$) then $G$ is abelian (the Klein 4-group).
- If $|h| \in \{3, 5, 7, \ldots\}$ (i.e. $|G| \in \{6, 10, 14, \ldots\}$) then $Z(G) = 1$. 

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- If \(|h| \in \{4, 6, 8, \ldots\}\) (i.e. \(|G| \in \{8, 12, 16, \ldots\}\)) then \(Z(G) = \{1, h^{n/2}\}.
- If \(|h| = \infty\) (i.e. \(|G| = \infty\)) then once again we obtain \(Z(G) = 1\).

Note that the groups arising in this question, are the dihedral groups.

4. Let \(G = GL_n(\mathbb{C})\), the group of invertible complex \(n \times n\) matrices, and let \(T \leq G\) be the subgroup consisting of all diagonal matrices with nonzero diagonal entries. Show that \(T\) is a maximal abelian subgroup of \(G\), i.e. the only abelian subgroup of \(G\) containing \(T\), is \(T\) itself.

**Solution.** Let \(S \in G\) be a matrix commuting with every element of \(T\); we must show that \(S \in T\). By assumption, \(S\) commutes with the diagonal matrix

\[
D = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & n
\end{bmatrix} \in T.
\]

(All that we really require about \(D\), is that it is diagonal, with distinct diagonal entries.) From here, one can view the hypothesis \(SD = DS\) as a system of \(n^2\) linear equations in the \(n^2\) unknown entries of \(S\), from which it is easy to deduce the desired conclusion that all off-diagonal entries of \(S\) must vanish. A more enlightened argument (and one that does not require writing out the linear system) is to recall that the hypothesis \(SD = DS\) implies that every eigenspace of \(D\) is invariant under \(S\). But the eigenspaces of \(D\) are the coordinate axes \(\langle e_1 \rangle, \langle e_2 \rangle, \ldots, \langle e_n \rangle\) (where \(e_1, e_2, \ldots, e_n\) are the standard basis vectors). To say that \(Se_i \in \langle e_i \rangle\) means that \(S\) is diagonal, as required.