#1. Rather than pulling a rabbit out of a hat, let me approach this systematically. We expect that \( G(2,3,6) \) is a group of isometries of the Euclidean plane preserving a tiling of the plane by triangles with angles \( \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6} \). Here is a portion of that tiling:

Here \( u, v, w \) are reflections in the three sides of the triangle \( T \), and \( x = uv, y = wu, z = vw \), so that \( x^3 = y^2 = z^6 = 1 \) and \( z = (xy)^{-1} = y^2x \).

The group \( G = \langle x, y, z \rangle \) contains translations, and one is given by \( wvwvwu = z^{-2}y = xyxyx^2 \).

Without access to the full theory of groups of type \( G(l,m,n) \), we see that the group \( G \)
generated by the three rotations $x, y, z$ as above, contains a translation $xyxyz^2$, which has infinite order; hence $|G| = \infty$. Moreover since $x^3 = y^2 = z^6 = 1$, $G$ is a homomorphic image of $G(2,3,6) = \langle x, y, z : x^3 = y^2 = z^6 = xyz = 1 \rangle$. Thus $G(2,3,6)$ itself must be infinite.

(Remark: As claimed in class, $G$ is actually isomorphic to $G(2,3,6)$; i.e. the group generated by the three rotations $x, y, z$ has presentation given by $x^3 = y^2 = z^6 = xyz = 1$.)

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Coset enumeration shows that $[G:H] \leq 8$.

(b) Consider the elements $\sigma = (123), \tau = (1342)$ in $S_4$. It is not hard to see that $\langle \sigma, \tau \rangle = S_4$. (By Lagrange's Theorem, $\langle \sigma, \tau \rangle$ is a subgroup of order
divisible by 12, and since \( \tau \notin A_4 \) this forces \( <0, \tau> = S_4 \). Here we may use the fact that
\( S_4 \) has only 4 normal subgroups: 1, \( S_4 \), \( A_4 \) and the Klein 4-group \( <(12)(34), (13)(24)> \); in particular \( A_4 < S_4 \) is the unique subgroup of order 12. Also \( \sigma^3 = \tau^4 = (\sigma \tau)^2 = 1 \), so \( S_4 \) is a homomorphic image of \( G \). But since
\( |G| = [G: H][H] \leq 8 \cdot 3 = 24 \), this forces \( G \cong S_4 \).

3. The abelianization of \( F_n \) (isomorphic to \( F_n/[F_n, F_n] \)) is a free abelian group on \( n \) generators, isomorphic to \( \mathbb{Z}^n \). Note that the minimum number of generators of \( \mathbb{Z}^n \) is \( n \).

(Any generating set for the additive group \( \mathbb{Z}^n \) is in particular a spanning set for the vector space \( \mathbb{Q}^n \) over \( \mathbb{Q} \); such a set has at least \( n \) elements, by linear algebra.) The groups \( \mathbb{Z}^n \) and \( \mathbb{Z}^m \) are not isomorphic if \( n \neq m \) (since the minimum number of generators is not the same), so \( F_n \) \( \neq F^m \).

4. As shown in class (I think?),
\( S_4 \cong <x, y, z : x^2 = y^2 = z^2 = (xy)^3 = (xz)^2 = (yz)^3 = 1> \)
(via \( x \mapsto (12), y \mapsto (23), z \mapsto (34) \). )
Look for three elements of $\text{PGL}_2(3)$ that permute $U_1 = \langle (0) \rangle$, $U_2 = \langle (1) \rangle$, $U_3 = \langle (2) \rangle$, $U_4 = \langle (3) \rangle$ in the same way:

\[ X = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} : \begin{array}{l}
U_1 \leftrightarrow U_2 \\
U_3 \rightarrow U_3 \\
U_4 \rightarrow U_4
\end{array} \]

\[ Y = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} : \begin{array}{l}
U_1 \rightarrow U_1 \\
U_2 \leftrightarrow U_3 \\
U_4 \rightarrow U_4
\end{array} \]

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} : \begin{array}{l}
U_1 \rightarrow U_1 \\
U_2 \rightarrow U_2 \\
U_3 \leftrightarrow U_4
\end{array} \]

We check that $X^2 = Y^2 = Z^2 = (XY)^3 = (XZ)^2 = (YZ)^3 = I$ as elements of $\text{PGL}_2(3)$ (note: elements of $\text{PGL}_2(3)$ are matrices up to nonzero scalar multiple) so $\langle X, Y, Z \rangle$ is a subgroup of $\text{PGL}_2(3)$ which is a homomorphic image of $S_4$.

But the group $\langle X, Y, Z \rangle$ induces all $4! = 24$ permutations of $U_1, U_2, U_3, U_4$ so $|\langle X, Y, Z \rangle| = 24$.

It follows that the epimorphism $S_4 \rightarrow \langle X, Y, Z \rangle$ is in fact an isomorphism. Since $|\text{PGL}_2(3)| = 3(3^2-1) = 24$, we have $S_4 \cong \text{PGL}_2(3)$. 
#5. Let $S = (1, 0), \ T = (0, 1), \ U = (ST)^2 = (0, 2)$. Clearly $S^3 = T^3 = U^2 = I$ and $SU = US, \ TU = UT$.

Using GAP, we find that the group

$$G = \langle s, t, u : u = (st)^2, \ s^3 = t^3 = u^2 = 1, \ su = us \rangle$$

has order 24. It is not hard to verify that $\langle S, T, U \rangle = SL_2(3)$. (Note that $\langle U \rangle = Z(SL_2(3))$ so it suffices to check that the images of $S$ and $T$ modulo $\langle U \rangle$ generate $SL_2(3)/\langle U \rangle = PSL_2(3) \cong A_4$.

The latter isomorphism follows from #4; and the images of $S$ and $T$ mod $\langle U \rangle$ give two non-commuting 3-cycles in $A_4$, such as $(123)$ and $(234)$; in this context it is easy to see that $\langle (123), (234) \rangle = A_4$ as required.)

Now $SL_2(3)$ is a homomorphic image of $G$; but $|G| = |SL_2(3)| = 24$ so we have $SL_2(3) \cong G$ as required.

As an alternative to the three generators listed above, we can solve for $u = (st)^2$ to obtain also

$$SL_2(3) \cong G = \langle s, t : s^3 = t^3 = (st)^4 = 1, \ s(st)^2 = (st)^2 s \rangle.$$