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**INVITED COMMENTARY FOR THOMPSON’S PAPER AND PRESENTATION**

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**Introduction**

Consider an 8th-grade student, Ally, who is about to embark on a unit studying quadratic functions through comparisons of the heights, lengths, and areas of rectangles that grow larger while maintaining their height/length ratios. Curious about Ally’s conceptions of area, I asked her to draw a rectangle with an area of 24 square feet, to label the length and the width, and to show where the 24 would be in her drawing. Ally produced the following figure:
Ally knew that she needed to label a rectangle in which the value of the height multiplied by the value of the length would produce 24. The numbers 4 and 6 worked numerically, but Ally’s diagram suggests that she did not see 4 as a unit of measure of the height of the rectangle. Moreover, Ally generated 24 rectangles, but it is unclear whether or how she saw the 24 rectangles relating to the heights of 4 feet and 6 feet. It seems likely that Ally, like Simon’s pre-service teachers (Simon & Blume, 1994), saw the area of the large rectangle as a region that needed to be covered by chunks of area (in this case the 24 small rectangles). For Ally, area was not composed of two lengths combined multiplicatively, and multiplication was not a quantification of area.

This example emphasizes one of Thompson’s (2011) key points: that quantities are mental constructions, and that their creation is often effortful for students. I have come to believe that understanding students’ conceptions of quantities and acts of quantification is one of the key challenges in mathematics education. Instructional approaches typically take quantities such as length, area, and perimeter as self-evident. Students’ conceptions of these attributes, however, may be quite different from the conceptions held by their teachers or embedded in textbook treatments. The failure to attend to the dialectic between students’ conceptualizing a quantity and conceptualizing a quantification of it can lead to substantial obstacles for students’ mathematical understanding.

What are the potential implications of shifting to a way of thinking about mathematics instruction that emphasizes, as a core activity, the acts of quantifying and reasoning with relationships between quantities? Thompson describes a number of implications for students’ understanding of algebra, for students’ learning of calculus and differential equations, and for modeling, generalization, and transfer. I will discuss these briefly and then propose a final implication of a quantitative-reasoning focus in mathematics, namely, its potential to influence students’ conceptions of proof.

Thompson describes four dispositions that can support students’ creation of algebra from quantitative reasoning: A disposition to represent calculations, a disposition to propagate information, a disposition to think with abstract units, and a disposition to reason with magnitudes. Constructing quantities and quantification are important mathematical activities, but further embedding reasoning within more complex quantitative situations can necessitate (Harel, 2008) the development of these dispositions to support algebraic reasoning. For instance, let us return to Ally and her classmates two weeks into their quadratic functions unit (for more detail, see Ellis, 2011a). As the students worked with different sized rectangles that grew proportionally, they developed the ability to predict the area from the height of a given rectangle. For instance, a 2 cm by 9 cm rectangle grows in such a way that for every 1-cm increase in height, the length
gains 4.5 cm: or, as one student described it, “It would go over 4.5 for every time you go up the height 1.” The student determined that he could find the area by multiplying the length (4.5\(h\)) by the height (\(h\)), and wrote \(A = 4.5h^2\). For a general \(H\) cm by \(L\) cm rectangle, the students were able to express the area as \(A = aH^2\), where \(a\) represents the ratio of the increase in length to the increase in height. One student, Tai, offered the following description: “[I]t is always the difference in length divided by the difference in height.” Another student, Daeshim, formalized this relationship as “\(a = \frac{dL}{dH}\)”.

In this example we see evidence of the disposition to write formulas (e.g., \(A = 4.5h^2\)) for evaluating a quantity (area). We see the disposition to propagate information: Given information about a growing rectangle’s area and height, the students could determine the length for any given height, and therefore determine its area. Moreover, the students were positioned to think about the role of the parameter \(a\) in \(A = aH^2\) in terms of how the quantities height and length covaried, reasoning with abstract units. Students also demonstrated the disposition to reason with magnitudes, comparing both the size of a rectangle’s height to its length (Tai explained that the length was 4.5 times as big as the height) as well as comparing the increase in the rectangle’s height to the increase in its length as it grew (“It would go over 4.5 for every time you go up the height 1.”) Embedding student reasoning within problems that necessitate the construction of complex quantitative relationships can, I suspect, offer a strong foundation not only for algebraic reasoning but also for an understanding of function in particular. As I discuss in a moment, building quantitative relationships can support a function understanding from a covariation perspective, but it can also serve as a support for a flexible understanding of function that enables shifts between the covariation perspective and the correspondence perspective. The students in the quadratic functions unit relied on their images of changing heights, lengths, and areas to construct formulas such as \(A = 4.5h^2\) and to make sense of these formulas both in terms of how height and length covary and how to directly determine area from the height. As Jim described, “[\(h\)] can be any value, times 4.5 is your length, times \(h\) again…is your area.”

Thompson discusses the role of quantitative reasoning in building covariation, and the ways in which this can support the development of ideas foundational to calculus and differential equations. As he points out, the operations that make up covariational reasoning are the same ones that enable the construction of invariant relationships among quantities in dynamic situations. Being able to imagine the length of a rectangle as 4.5 times as big as the height regardless of the size of the rectangle, as both the height and the length grow together, supports the operations for envisioning varying magnitudes. As Thompson notes, “This act, of unifying two quantities conceptually within an image of a situation that changes while staying the same, is nontrivial. Yet it is at the heart of using mathematics to model dynamic systems.” (p. 26) As we saw with Castillo-Garsow’s (2010) work and Moore’s (2010) work, the ways in which students imagine how quantities’ values vary has profound implications for conceiving situations that embody variation.

One can witness in the above example the students’ propensity to create generalizations about the relationships between the heights, lengths, and areas of growing rectangles. This recent work supports earlier findings that reasoning with quantitative relationships can alter the character of students’ generalizations; students focusing on quantities demonstrate the propensity to generalize about dynamic relationships rather than patterns, procedures, or rules (Ellis, 2007). This also suggests that attending to mathematical modeling from a quantitative-reasoning perspective can support more productive generalizations about relationships between dynamic phenomena, which in turn can foster the development of function understanding in algebra and provide a foundation for reasoning about ideas in calculus.

I would now like to suggest a final implication of a quantitative reasoning approach to
mathematics, which is its potential for supporting deductive proof. Based on students’ work and historical development, Harel and Sowder (1998; Harel, 2007) proposed a taxonomy of proof schemes consisting of three classes: External conviction, Empirical, and Deductive. The deductive proof scheme class consists of two categories: The transformational proof scheme category and the modern axiomatic proof scheme category. The latter is not generally accessible to students, particularly at the K-12 level, but the transformational proof scheme category is Harel and Sowder’s description of how students can validate or refute conjectures by means of logical deduction. Within the transformational proof scheme, students employ goal-oriented operations on mental objects and anticipate the results of those operations. They must be capable of transforming images by means of deduction. Three criteria are employed to determine if a student exhibits the transformational proof scheme: (a) a student must consider the generality aspects of a conjecture; (b) a student must apply goal-oriented, anticipatory mental operations; and (c) a student must transform images as part of his or her deduction process.

As Thompson (2011) describes, the mathematics of variation involves imagining a quantity whose value varies. Doing so requires anticipating its measure having different values at different moments in time. The operations involved in conceptualizing a varying quantity are anticipatory and goal-oriented. Moreover, they require the transformation of images as students imagine variation: “...in thinking of the magnitude varying, we imagine it varying in microscopic bits, each bit itself entailing variation upon close inspection.” (p. 24) These operations are the very operations that students must employ in order to reason within the transformational proof scheme.

Returning one last time to the students in the quadratic functions unit, consider Daeshim’s justification that the second differences in a table with uniform increments for a quadratic function $y = ax^2$ are constant (Ellis, 2011b). In particular, he argued that the second differences, which he conceptualized as the difference in the rate of growth (DiRoG) of the area of the rectangle, would amount to twice the area of the original H by L rectangle when it began to grow:

$$\text{DiRoG} = \frac{\text{Original height of rectangle}}{\text{Original length of rectangle}} \times 2$$

![Figure 2: Daeshim’s justification that the DiRoG is twice the area of the H by L rectangle](image)

Daeshim explained that because he could calculate the difference in the rate of growth of the area as 2HL each time the rectangle grows an additional H units in height and L units in length, the DiRoG must be twice the original area of the rectangle. Daeshim’s justification certainly has limitations; for instance, his drawing only addresses a particular type of growth in which the height and length increase by whole-unit increments of H and L. However, despite its flaws, Daeshim’s justification suggests he was operating within the transformational proof scheme. He drew a general H by L rectangle rather than relying on a specific example. He relied on logical inferences. He also applied goal-oriented operations as he anticipated the relationships between the area of the rectangle and its original height and length each time it grew; Daeshim could imagine the rectangle growing, albeit in discrete jumps.

**Conclusion**

Focusing on quantitative relationships supports reasoning that is flexible and general in character but does not necessarily rely on symbolic expressions (Smith & Thompson, 2008). This is the type of reasoning that can support the development of the transformational proof scheme. Moreover, because the nature of students’ generalizations when reasoning with quantitative relationships is closely tied to the important mathematical structures embedded in problems, students may be better able to exploit their understanding of these structural relationships when creating proofs. The transition from informal reasoning to more formal, deductive reasoning is not well understood; our field lacks studies addressing ways to help students make this transition and clarifying the mechanisms underlying it. I propose that a quantitative-reasoning approach offers a great deal of promise for developing and supporting students’ abilities to produce deductive proofs.

**References**


