RELATING ONE AND TWO-DIMENSIONAL QUANTITIES:
STUDENTS’ MULTIPLICATIVE REASONING IN
COMBINATORIAL AND SPATIAL CONTEXTS

Erik S. Tillema
Indiana University Purdue University Indianapolis

Abstract
In this paper, I outline the distinction that Vergnaud (1983) made between isomorphism of measures problems and product of measures problems. I use this distinction as a rationale for considering how the multiplicative structures that three eighth grade students used to solve Cartesian product problems could be a basis for understanding their solution of problems that involved relating lengths, widths, and areas. By doing so, I draw a connection between students’ combinatorial reasoning and how they quantified situations that involved their spatial and multiplicative reasoning.

Vergnaud (1983) provided a mathematical analysis of the multiplicative conceptual field in which he identified how the multiplicative reasoning in many combinatorics problems was similar to the multiplicative reasoning involved in establishing quantitative relationships among lengths, areas, and volumes (i.e., relationships among geometric quantities). Although he posited that there was a connection between these two domains of multiplicative reasoning, the connection that he proposed was based on a mathematical analysis of problem situations not based on making models of students’ reasoning. In this paper, I outline the mathematical distinctions that led Vergnaud to draw his conclusions. I then provide a summary of data from a 3-year constructivist teaching experiment. In this summary, I first outline four different levels of a scheme that three eighth grade students constructed while solving Cartesian product problems. After I have outlined these different levels of the scheme, I provide an example of how students who were operating at the third and fourth level of the scheme quantified situations that involved lengths and areas differently.

Vergnaud
As part of Vergnaud’s (1983) analysis of the multiplicative conceptual field he differentiated between isomorphism of measures problems and product of measures problems. He defined isomorphism of measures problems as any problem that from an adult’s mathematical perspective can involve a direct proportion between two measure spaces. These problems include equal groups problems, partitive and quotitive division problems, ratio problems, direct proportion problems, and constant rate problems (see also, Van Dooren, De Bock, Janssens, & Verschaffel, 2008). One example of an isomorphism of measures problem is an equal groups problem like the Donut Problem.

The donut problem, There are 6 donuts in a package. Jim buys 4 packages of donuts. How many donuts did Jim buy?

It is an isomorphism of measures problem because it can be seen as involving two measure spaces, the number of donuts and the number of packages, and its solution can involve establishing a direct proportion between these two measure spaces (i.e., 6 donuts per 1 package = 24 donuts per 4 packages).
He contrasted isomorphism of measures problems with product of measures problems which are any problem that from an adult’s mathematical perspective can involve a Cartesian product between two measure spaces into a third measure space (Vergnaud, 1983, p. 134). The Two-Suits Card Problem (a Cartesian product problem) illustrates one type of problem that is a product of measures problem.

**The two-suit card problem.** You have the Ace thru King of Hearts—13 cards. Your friend has the Ace thru King of Clubs—13 cards. You make two-card hands by pairing one of your cards with one of your friend’s cards. How many different two-card hands can you make?

The Two-Suit Card Problem involves a Cartesian product between two measure spaces into a third—the first measure space is the number of hearts, the second is the number of clubs, and the third is the number of two-card hands. Because the problems involve three measure spaces (as opposed to only two), the basic properties of the units in these problems differ from the basic properties of the units in isomorphism of measures problems. That is, the measure of the units of the product, two-card hands, is different from the measure of either of the two quantities that are used to create the product, hearts and clubs. This differs from an isomorphism of measures problem like the Donut Problem where the measure of the product is donuts, which is the same as the measure of one of the quantities that is used to produce this product. This difference occurs because the basic units that are counted, two card hands, can involve the property that \(1 \times 1 = 1\), one heart and one club creates one two-card hand (Behr, Harel, Lesh & Post, 1994; Confrey, 1994; Nesher, 1988; Nunes & Bryant, 1996; Outhred, 1996). This property means that the creation of one two-card hand involves establishing a multiplicative relationship that is not involved in creating one donut, which are the basic units that are quantified in the Donut Problem.

In addition to these properties, Vergnaud (1983) also identified that product of measures problems do not involve a direct proportion. Instead they involve two proportions that are independent of each other, which Vergnaud called a double proportion. The Two-Suit Card Problem involves a double proportion because the number of two-card hands is proportional to the number of hearts and is also proportional to the number of clubs (in this case the proportions are equal). Moreover, each proportion is independent of the other because the number of hearts can be varied independently of the number of clubs.

Here Vergnaud intended the term *double* proportion to be closely related to problems that involve establishing one quantity as proportional to the square of another quantity. However, he used the term *double* proportion (as opposed to square proportion) because it can be applied to a more general class of problems. The Ice Cream Problem, an arrangement problem, helps to illustrate this point.

**The ice cream problem.** There are 8 ice cream flavors. How many two-scoop ice cream cones can you make if the order of the scoops matters?

In the Ice Cream problem, the number of two-scoop cones can be considered proportional to the square of the number of flavors. The number of two-scoop cones can be considered proportional to the square of the number of flavors because there is only one quantity, the number of flavors, that is used to create the number of two-scoop cones. For this reason, tripling the number of flavors means that there are three times as many flavors for the first scoop and three times as many flavors for the second scope, which produces nine times as many two-scoop cones. Thus, the primary difference between a problem that can involve a square proportion and one that can involve a double proportion is whether a problem is conceptualized as having one quantity (a square proportion) or two quantities that can be varied independently of each other (a double proportion).
All of the mathematical properties that I have outlined so far pertain to problems that involve finding the area of a rectangle given its length and width. That is, finding the area of a rectangle given its length and width can be conceptualized as involving three measure spaces—the units used to measure length, the units used to measure width, and the square units used to measure area. The square units used to measure area are different from either of the units used to measure length and width, and the formation of square units involves the property that \(1 \times 1 = 1\) (one unit of length and one unit of width creates one square unit of area). Moreover, the length and width of a rectangle are each proportional to the area, and each proportion can be varied independently of the other, and so this kind of problem can be considered to involve a double proportion.\(^1\) In the case when a problem involves finding the area of a square, then its area is proportional to the square of the side length because there is only one quantity, the side length, which can be varied (an observation that is similar to my analysis of the arrangement problem). For all of these reasons, Vergnaud considered problems that involved establishing a multiplicative relationship among length, width, and area to be product of measures problems too.

So Vergnaud's analysis indicates that from a mathematical perspective there is a similarity between the multiplicative reasoning involved in solving combinatorics problems and the multiplicative reasoning involved in establishing multiplicative relationships among lengths, widths, and areas.\(^2\) Such a connection suggests that the operations and schemes that students construct to solve combinatorics problems have the potential to be related to the relationships they establish among one and two-dimensional quantities. I develop this idea by summarizing the results of a study that I conducted with middle grades students, and connect the results of these studies to implications for how students are likely to establish relationships among one and two-dimensional quantities.

The Study

Overview of the Study

The study involved three eighth grade students who were participants in a 3-year teaching experiment that was conducted during their sixth, seventh, and eighth grade years. Prior to the students’ eighth grade year, they had solved problems that from my perspective were isomorphism of measures problems (i.e., problems that could be conceived of as involving a direct proportion). To understand their multiplicative reasoning in these contexts, I used Steffe’s (1992, 1994) framework for multiplicative reasoning, which is a framework that has been developed largely in contexts that an adult would classify as isomorphism of measures problems (for an example, see Hackenberg & Tillema, 2009). In contrast, during the students’ eighth grade year I planned to have them solve a range of problems that from my perspective were product of measures problems (i.e., problems that could be conceived of as involving a double proportion). Therefore, the beginning of the students eighth grade year was a transition point in the ex-

\(^1\) In geometry problems that involve similar figures the length and width are not considered independent variables because to produce a similar figure requires multiplying both by the same scale factor. When the length and width are multiplied by the same scale factor, the area of the similar figure is proportional to the square of the scale factor.

\(^2\) Although I do not discuss it here, Vergnaud’s definition of product of measures problems relies on the binary and recursive nature of multiplication. That is, Vergnaud considered Cartesian product problems that involve three quantities (e.g., adding the Ace through King of diamonds to the Two-Suit Card Problem to turn it into a Three-Suit Card Problem) to adhere to all of the properties of product of measures problems: they can be solved by forming a Cartesian product between two of the quantities, and then a second Cartesian product between the result of the first Cartesian product and the third quantity.
periment—the students had solved primarily isomorphism of measures problem until that point after which the students solved a range of problems that were product of measures problems.

I began the students’ eighth grade year by presenting them with Cartesian product problems like the Two-Suit Card Problem in order to establish baseline data on how the students reasoned about product of measures problems. As part of establishing baseline data on how the students reasoned, I identified four levels of a scheme that they used to solve these problems. The first and second levels of the scheme were transitional in the sense that the students produced them only because product of measures problems were novel to them—they did not continue to operate in these ways during the remainder of the experiment. In contrast, the third and fourth levels of the scheme helped me to understand differences in how the students quantified a range of situations that could be characterized as product of measures problems.

Before outlining the different levels of the scheme, I first give a brief overview of scheme theory, and then I outline Steffe’s (1992, 1994) framework for multiplicative reasoning—the framework that I had used to understand the students’ multiplicative reasoning prior to the students eighth grade year. I outline Steffe’s framework for two reasons: first, it was the starting place for developing the different levels of the scheme that students’ constructed for solving Cartesian product problems; and second, it remained a useful tool in interpreting how students quantified situations that were I deemed were product of measures problems.

**Scheme Theory**

A *scheme* has three parts—an assimilatory mechanism, an activity, and a result (von Glasersfeld, 1995). The assimilatory mechanism involves a person in making an interpretation of a problem. The assimilatory mechanism triggers the activity of a scheme, in which a person carries out mental operations on physical or mental material or both. The activity, then, produces a result. When a person can use the result of a scheme in assimilation of a future problem without carrying out the activity that produces it, the person has *interiorized* the scheme and constructed a concept (von Glasersfeld, 1995).

**Steffe’s Framework for Students’ Multiplicative Reasoning**

Steffe (1992, 1994; see also, Steffe & Cobb, 1988) has identified three qualitatively distinct multiplicative concepts that students produce (cf. Mulligan & Mitchelmore, 1997). The multiplicative concepts depend on the level of interiorization of students’ iterating and units coordinating operations, where he has defined *iteration* as the repetition of a unit, and *units coordination* as the insertion of units within units (cf. Confrey, 1994; Kamii & Housman, 2001). To illustrate the different multiplicative concepts I give an example of the kind of reasoning that students who use each concept are likely to produce in their solution of an equal groups problem like the Donut Problem (stated earlier).

Students who use the first multiplicative concept (MC1) can solve the Donut Problem by iterating a unit of one 6 times (i.e. a donut six times), and engaging in a units coordination by inserting the 6 units into a containing unit (i.e., a package). When students operate in this way, Steffe (1992) has called the unit structure that they create a unit of units in *activity* because they create this unit structure as part of their activity to solve the problem (i.e., they create a composite unit in activity). To continue solving the Donut Problem, an MC1 student is likely to continue iterating a unit of one and after 6 iterations insert these units into a containing unit to create another unit of 6 units in activity, repeating these operations until they have created 4 units of 6 units in activity.

Students who use the second multiplicative concept (MC2) have interiorized these operations, and so they can assimilate problems like the Donut Problem using a composite unit (i.e.,
a unit of units structure). MC2 students can also iterate a composite unit, and can operate on a composite unit as part of their solution. This means that they can reason that 6 and 6 is 12 because 6 and 4 is 10, and 10 and 2 is 12. By doing so they iterated a unit of 6 units twice, and they operated on the second unit of 6 units by disembedding a unit of 4 units and a unit of 2 units to strategically combine these parts with the first unit of 6 units. A disembedding operation means that students establish parts of the composite unit (in this case a unit of 4 units and a unit of 2 units) as independent of the composite unit, while also maintaining the composite unit as a whole (in this case a unit of 6 units) (Steffe, 1992). To finish their solution of the Donut Problem, MC2 students are likely to continue iterating a composite unit and using strategic reasoning to combine composite units. When MC2 students produce the final result, 24, they may insert this unit structure into a containing unit to create 24 as a unit of 4 units of 6 units. If they engage in this units coordination, they do it as part of their activity and so they create a unit of units in activity.

Students who use the third multiplicative concept (MC3) have interiorized the operations just described, which enables them to assimilate problems using a unit of unit of units structure. The interiorization of these operations enables them to solve an extension of the Donut Problem that involves 12 packages of 6 donuts by reasoning that 10 packages of 6 donuts is 60 and 2 packages of 6 donuts is 12 so the total number of donuts is 72. That is, they no longer need to engage in iterating or units coordinating operations to solve an equal groups problem. Instead they can operate on a unit of 12 units of 6 units by disembedding a unit of 10 units of 6 units and a unit of 2 units of 6 units (i.e., treat each of these parts as independent of the unit of 12 units of 6 units while maintaining the unit of 12 units of 6 units as a whole), and they use multiplication to evaluate each of these three level of unit structures to determine that the result is 72.

**Modifying Steffe’s Framework to Account for Students’ Solution of Product of Measures Problems**

The three eighth grade students that helped me to modify Steffe’s framework entered their eighth grade year as MC3 students—they could assimilate many types of problems using a unit of units structure, and they could operate on this structure (for examples, see Hackenberg & Tillema, 2009; Hackenberg, 2007, 2010). However, their solution of Cartesian product problems involved them in making changes to all three parts of their scheme for whole number multiplication, and these modification led to my identification of four levels of the scheme (Tillema, under review a).

**Level 1.** At the first level of the scheme students assimilated Cartesian product problems using two unit of units structures (i.e., composite units) simultaneously. In the case of the Two-Suits Card Problem (given earlier), the two composite units were thirteen hearts and thirteen clubs. The activity of their scheme involved two novel operations—ordering and pairing—and also involved a third operation, iteration, which I defined as part of outlining Steffe’s framework. I defined an ordering operation as any time a student supplied a property that enabled them to create an order for the units of the composite units they used in assimilation. For example, in the Two-Suits Card Problem students ordered the hearts when they used the digit on each card to order the hearts. I defined a pairing operation as any time students created a correspondence between one unit of each composite unit and applied their unitizing operation to this correspondence. For example, in the Two-Suit Card Problem students paired the two of hearts with the two of clubs when they created a correspondence between these two cards and applied their unitizing operation to this correspondence to create a two-card hand. I called the units that resulted from this operation (i.e., the two-card hands) pairs because they contained two units, but were counted as a single entity.
Initially to solve problems like the Two-Suits Card Problem students ordered the units of each composite unit, paired the first unit of one composite unit with all of the units of the other composite unit, and then continued these pairing operations with the second unit of one of the composite units. For example, a student would pair the two of hearts with the two of clubs, the two of hearts with the three of clubs, the two of hearts with the four of clubs, etc. They continued pairing the two of hearts with each club, and then when they had paired the two of hearts with each club, they engaged in similar operations with the three of hearts—pairing it individually with each club. So the students iterated their pairing operation to produce the pairs, and they followed a lexicographical ordering to produce the two-card hands: they paired the first heart with the first club prior to pairing the first heart with any of the other clubs (i.e., pair (1,1) was before pair (1, 2), (1, 3), etc.), and they also paired the first heart with each of the clubs prior to pairing the second heart with any of the clubs (i.e., (1,1) thru (1, 13) were created before (2, 1)) (see English’s odometer method, 1991, 1993). At this level, the students counted the pairs they created by ones, which I took as indication that they treated them as individual pairs. This also meant that the students did not create a multiplicative relationship between the pairs that they produced as a result of their scheme and the composite units they used in assimilation. I took this as indication that producing the pairs consumed the students’ awareness and the composite units they used in assimilation were no longer part of the situation when the student was done solving the problem.

**Level 2.** At the second level of the scheme, the students assimilated the problems in a similar manner—using two composite units. However, the activity of their scheme consisted of five operations—ordering, disembedding, pairing, iterating, and units coordinating. Michael’s solution of the Coin-Die Problem illustrates this level of the scheme. To set up the Coin-Die Problem, I had Michael flip a coin and his partner roll a die with the intent of asking them how many different outcomes they could get together if they repeated this experiment. Prior to asking them this question, Michael did the following.

M: Wait a minute. [M writes down “T” and “H” on his piece of paper, and “1”, “2”, “3”, “4”, “5”, “6”. He sweeps his pencil from “T” to where he has written “1”, from “T” to where he has written “2”, and then from “T” to where he has written “3”.] It’s like I said the other day. So, it’s that [points to where he has written “T”] times that [points to where he has written “1”, “2”, “3”, “4”, “5”, “6”], and then that [points to where he has written “H”] times that [points to where he has written “1”, “2”, “3”, “4”, “5”, “6”], and then times both of those or something like that. [He sweeps his pencil two more times through the air.]

T [to encourage him to continue]: Mhmmm, yeah.

M: Thirty-six.

T: Thirty-six?

M [draws a line between “T” and each of the numbers, and then draws a line between “H” and each of the numbers, recording each outcome in a list as “T1”, “T2”, etc. During this time, the teacher has a conversation with Carlos who is already done solving the problem.]
T [Michael finishes drawing his lines]: How many did you get total?

M: It’d be twelve.

T: Why would it be twelve?

M: Cause there is six [points to where “T” is written on his paper] and six [points to where “H” is written on his paper].

His subsequent statements were interesting because they indicate that he could not predict ahead of time the total number of pairs he was going to produce: he wrongly predicted that there would be “thirty-six”. When I threw doubt on his response by saying, “Thirty-six?”, he began to draw lines between “H” and each number he had written, and then “T” and each number he had written, and created a list for each coin-die outcome that he created. Once he had created all of the coin-die pairs, he said he had created “six (pointing to ‘H’) and six (pointing to ‘T’)”, indicating that he treated the pairs as composites as opposed to as individual pairs. Here I interpret these actions as indication that he disembedded the first unit of the unit of two units (the heads), the first unit of the unit of 6 units (the one on the die), and paired them to make the first coin-die pair. He then iterated these operations to produce all of the coin-die pairs with heads, at which point he inserted them into a containing unit to create a unit of pairs in activity (i.e., one set of six pairs as opposed to six individual pairs). He then operated similarly to create the coin-die pairs with tails.

The fact that students used their disembedding and units coordinating operations at this level enabled them to establish a multiplicative relationship among three quantities: a unit of one (in this case, the heads on the coin), a unit of units (in this case, the six numbers on the die), and a unit of pairs (in this case, the six coin-die pairs) (Figure 1). Using their disembedding operation is what enabled them to retain the unit of one and the unit of units as part of the situation and what enabled them to produce the pairs independently of these other two quantities. However, the result of the situation was a multiplicative relationship among only one of the units of one composite unit (i.e., not all of the units of this composite unit), all of the units of the other composite unit, and the unit of pairs that they created as part of their activity. That is, all but one of the units from one of the composite units dropped away from the situation as part of creating this relationship. The student could return to the composite units they used in assimilation to create the multiplicative relationship again, as Michael did in the coin die problem when he created the coin-die pairs with tails, but each time the students created this multiplicative relationship it was only among a unit of one (not a unit of units), a unit of units, and a unit of pairs. I make this inference based on the fact that students at this level carried out their pairing operations between each unit of one composite unit (e.g., heads and tails) and all of the units of the other composite unit (e.g., the numbers on the die), which indicates that they had to engage in these operations.

Figure 1. Creating a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs in activity
**Level 3.** At this level of the scheme, the students assimilated situations using the multiplicative relationship that was a result of their scheme at the second level, and they iterated this relationship in order to produce the result. Carlos’s solution of the Outfits Problem illustrates this level of the scheme.

Data Excerpt 3: Carlos’s Solution of the Outfits Problem

T: At home, you have four shirts and three pants. How many outfits could you make?

C [The teacher-researcher poses the Outfits Problem to Carlos. Carlos sits in concentration for four seconds]: Twelve.

T: Great! How’d you know that?

C: Cause you multiply four times three and that equals twelve.

T: Could you draw me a picture to show me why it’s four times three?

C [identifies a color for each shirt and a color for each pants and makes a list for all of the outfits. As he finishes his list, he says]: So you count all these up and you get… there’s three here, three here, three here, and three here. You add these two up you get six. You add these two up you get six. You add these two and you get twelve.

T: Alright, tell me a little bit about what you have here [pointing to Carlos’s list]. Will you explain that to me?

C: Since its four shirts and three pants, then you have one shirt for each of three pairs of pants, and then it’s the same for the others so when you put them all together you get twelve.

When Carlos wrote his list, it looked as if he carried out his pairing operation to make each outfit. However, the fact that he correctly identified the multiplication problem before he operated in the situation along with his statement, “you have one shirt for each of three pairs of pants, and then it’s the same for the others” do not support this interpretation. That is, he made no reference to making individual outfits. Therefore, I interpret this statement as indication that he assimilated the situation using a multiplicative relationship among a unit of one unit (e.g., a shirt), a unit of 3 units (i.e., the 3 pants), and a unit of 3 pairs (i.e., the 3 outfits he could make with one shirt). His statement, “there’s three here, three here, three here, and three here”, provides indication that he iterated that relationship four times to produce all of the outfits (Figure 2). The result, then, of his scheme was four iterations of the multiplicative relationship among a unit of one (a shirt), a unit of 3 units (the pants), and a unit of 3 pairs (the outfits).

However, students at this level did not engage in any further units coordinating activity. That is, they did not, for example, in Carlos’s case insert the four iterations of a unit of 1, a unit of 3 units, and a unit of 3 pairs (Figure 2) into a containing unit to establish a multiplicative relationship among a unit of 4 units, a unit of 3 units, and a unit of 4 units of 3 pairs (Figure 3). In fact, it was unclear from this study what might create the occasion for a student to engage in this units coordination. Nonetheless, it was clear that some students did because I could attribute to two of the students the interiorization of this multiplicative relationship, which is what constituted the fourth level of the scheme.

**Level 4.** It was not possible to determine whether students had interiorized a relationship among a unit of units, a unit of units, and a unit of units of pairs simply based on their solution of Cartesian product problems whose statement referred to two quantities (e.g., shirts and
Figure 2. Iterating a unit of one, a unit of units and a unit of pairs four times

Figure 3. A multiplicative relationship among a unit of units, a unit of units and a unit of units of pairs.

pants). Therefore, I give an example of Deborah’s solution of the Candy Problem whose set up and statement involved three quantities. To set up the Candy Problem, I had Deborah put four different colored candies into each of three bags. I then asked her to perform an experiment: to draw one candy out of each bag and record the three-color combination that she got. On her first turn, she drew a red candy out of the first bag, an orange candy out of the second bag, and an orange candy out of the third bag, and recorded the result on her paper as “ROO”. The following exchange then took place between Deborah and me.

3 From an adult’s perspective, this problem could be considered an arrangement problem because each bag contained candies with the same colors. However, Deborah did not consider it as such: she did not consider ordered outcomes in her solution, and I did not intend for her to consider them in this problem.
Data Excerpt 5: Deborah’s solution of the Candy Problem

T: How many different three-color candy combinations could you make if you kept repeating that experiment?
D: Sixteen.
T: How did you get that?
D: Four times four. I mean no twelve.
T: Twelve? How did you get twelve?
D: Four times three bags.
T: Why don’t you go ahead and make another combination?
D: Red. [She pulls out a red candy from the first bag.] Orange. [She pulls out an orange candy from the second bag.]

T: Uh oh. Is it going to be the same one? [The teacher-researcher is referring to making the same three-color combination on this turn as she had made on her first turn.]
D [smiles]: Yellow. [She pulls out a yellow candy from the third bag. She writes “ROY” below where she has written “ROO”). Oh, you could get red orange orange, red orange yellow, red orange red, red orange pink. So, it would be four times four. It would be sixteen. Wait, it would be more than that. It would be like four times four times four.

T: Oh. Why is it that?
D: Cause there is four combinations in each bag and for each letter there is like four combinations and another four combinations. Cause there is three bags. [She calculates this multiplication problem at the teacher-researcher’s request.]

T: Could you represent all sixty-four using a diagram?
D [She produces Figure 4.]: And there would be four of those.

After Deborah made the first three-color combination “ROO”, she concluded that there would be a total of “sixteen”, and then that there would be “twelve” three-color combinations. I take these statements as indication that she was uncertain about how to solve a problem that began with three quantities: had the problem involved only two bags of candies, then the problem would involve “sixteen” two-color combinations, but since the statement of the problem involved three bags she looked for another way to compute the solution, “four (candies) times three bags”.

Deborah’s subsequent statements and actions provide indication that she assimilated the problem using a multiplicative relationship between a unit of 4 units (the 4 candies in one of the bags), a unit of 4 units (the 4 candies in another bag), and a unit of 4 units of 4 pairs (the 16 two-color combinations she could make with these candies), and operated on this structure to produce the result. That is, she said, “Oh, you could get red orange orange, red orange yellow, red orange red, red orange pink. So, it would be four times four.” I take this statement as indication
that she paired a unit of one (the color red), with the multiplicative relationship among a unit of 4 units, a unit of 4 units, and a unit of 4 units of 4 pairs. Doing so produced a multiplicative relationship among a unit of one (the color red from the first bag), a unit of 4 units (the candies in the second bag), a unit of 4 units (the candies in the third bag), and a unit of 4 units of 4 triples (16 three-color combinations), which were the “sixteen” three-color combinations she could make with red as the first color. Once she established this multiplicative relationship I infer she could envision iterating it four times, but that she did not actually need to carry these iterations out to conclude that there were sixty-four three-color combinations. I make this inference based on her concluding quite quickly that it would be “four times four times four”, and then making the tree diagram where the first color was red, Figure 4, and then stating, “And there would be four of those”. This data excerpt illustrates that Deborah could assimilate situations using a multiplicative relationship among a unit of units, a unit of units, and a unit of units of pairs, which is what constituted the fourth level of the scheme for solving Cartesian product problems.

**Figure 4.** Deborah’s notation

### Implications for How the Students Quantified Situations

**Involving Lengths, Widths, and Areas**

Across the students’ eighth grade year, I could consistently attribute to one of the students the interiorization of the multiplicative relationship among a unit of one, a unit of units, and a unit of pairs, which I identified as the third level of the scheme. For the the other two students, I could consistently attribute the interiorization of the multiplicative relationship among a unit of units, a unit of units, and a unit of units of pairs, which I identified at the fourth level of the scheme. This difference had a significant impact on how the students established relationships among lengths, widths, and areas as the experiment unfolded during their eighth grade year. Students who could assimilate situations using the multiplicative relationship among a unit of units, a unit of units, and a unit of units of pairs were able to take the multiplicative relationship among a length, a width, and area as given in a situation, and operate on this relationship further. For example, I asked Michael, one of these students, the following question.
The field problem. There is a field that is 80 yards in length and 80 yards in width. How many squares are contained in this field that have an area of 100 square yards? (Figure 5)

![Image of a grid with red and blue squares]

Figure 5. The 64 areas of 100 square units

Michael sat in concentration for approximately 15 seconds, and then stated that there would be 64. I infer that he was able to determine this quantitative relationship precisely because he assimilated the situation using a multiplicative relationship among a unit of 80 units, a unit of 80 units, and a unit of 80 units of 80 square units. My inference is that after assimilating the situation using this structure he operated on it by structuring the unit of 80 units of length as a unit of 8 units of 10 units, and structured the unit of 80 units of width as a unit of 8 units of 10 units, which enabled him to structure the area as a unit of 64 units of 100 square units.

In contrast, Carlos, who operated at the third level of the scheme, which corresponded to assimilating situations using a multiplicative relationship among a unit of one, a unit of units, and a unit of pairs could not a priori establish situations like the Field Problem as one that involved a multiplicative relationship among a composite unit of length, a composite unit of width, and all of the area units that could be produced with these two composite units. Rather, he could assimilate situations using one unit of length, a composite unit of width, and all of the area units that could be created from the single unit of length, and the composite unit of width. For example, he could assimilate the Field Problem using one unit of length, a unit of 80 units of width, and a unit of 80 square units. He could subsequently iterate this relationship (or envision iterating this relationship) 80 times to produce the total area, but he could not operate on the result of these actions exclusively mentally (i.e., in visualized imagination). This precluded him from mentally establishing the type of quantitative relationships that Michael did to solve the Field Problem.

4 Here I use area units as opposed to pairs because it was a context involving continuous as opposed to discrete quantities. However, in the context of this problem it is likely that these area units were quite similar to the pairs he created in discrete contexts—each area unit contained a unit of length and a unit of width and he established them as independent from the units of length and the units of width.
Concluding Remarks
The next steps of this work are to carefully trace across the students eighth grade year how the differences in the multiplicative concepts that they initially established in their solution of Cartesian product problems were expressed in a broader range of product of measures problems. The example above serves as one example of these consequences, but much more careful analysis is likely to outline the kinds of modifications that students using these different multiplicative concepts made in the context of solving product of measures problems. Such work promises to inform current discussion among mathematics educators about the relationship among students’ spatial and multiplicative reasoning (e.g., DeBock, Van Doren, Janssens, & Verschaffel, 2007).

References


