EXPERIMENTAL ERRORS

1. PREFACE

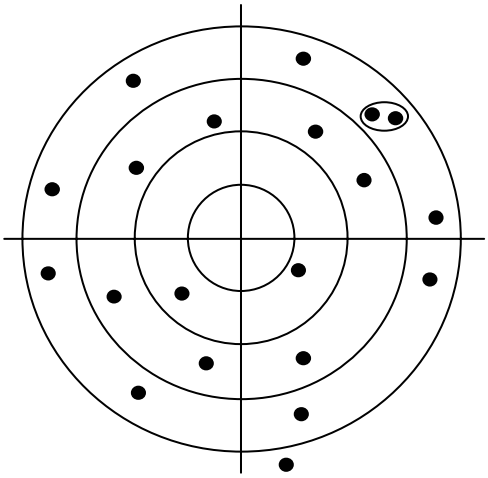
An experimenter must be concerned with the accuracy of the experimental results. Thus error estimation is an important aspect of experimental design, and an analysis of the errors in an experiment can involve very sophisticated mathematics. However, no level of mathematical sophistication can replace the use of common sense, shrewdness, and intuition. There are two kinds of errors, random and systematic, and the experimenter must recognize the sources of these errors and must devise strategies to minimize the impacts of these errors on the experimental results.

2. ACCURACY AND PRECISION

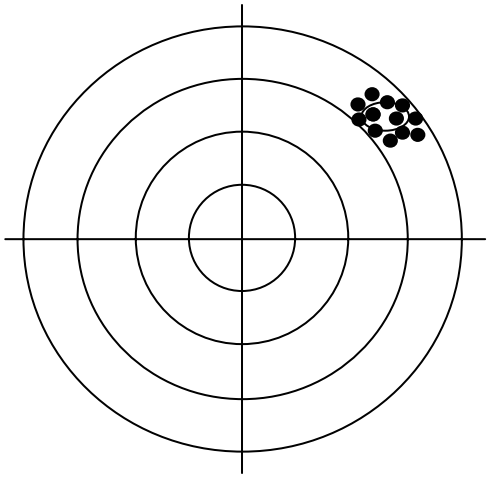
Although these two words are used in everyday life to mean essentially the same thing, they have quite different meanings in error analysis.

Accuracy, in its "technical" meaning, is a measure of how far an experimental value is from its "true" value. Precision, on the other hand, is a measure of the consistency among individual measurements of a quantity regardless of whether or not they are close to the "true" value. Thus one is said to have high precision if one's measurements are close to one another.

To illustrate these concepts, consider two people shooting at two targets. They end up with the following bullet holes in each target:



SHOOTER 1
Shaky hand, loose barrel, or gusty winds!
GOOD ACCURACY
BAD PRECISION



SHOOTER 2
Steady hand, consistent gun but steady winds, uncalibrated gun or wrong aiming
BAD ACCURACY
GOOD PRECISION

Figure 1. Accuracy and Precision Schematics

Shooter 1 had the bullets scattered over the entire target but on the average they (or rather their "center of

mass") seem to be centered on the target bull's eye. On the other hand, Shooter 2 placed all the bullets in a tight cluster. However, the cluster is off the bull's eye. Apparently the shooter and the gun are consistent but there must be an error some place, either with the gun or with the shooter's aim or with the wind.

In general, imprecise measurements are associated with random errors (Shooter 1) and inaccurate measurements are associated with systematic errors (Shooter 2). However, one must have a sufficient amount of data to decide whether one has a lack of precision or a lack of accuracy or a combination of both. Consider for example the two holes circled at each target. If each shooter had shot only two bullets and had ended up with these two holes, one would have been unable to say whether or not there is any difference between the two. But subsequent bullet holes reveal a large difference.

Accuracy and precision alone are concepts too vague to be a substitute for rigorous error analysis. Nonetheless they are useful concepts for identifying types of errors.

3. RANDOM ERRORS

The first and foremost problem with an experiment is that it is virtually impossible to obtain the "correct" result. This may sound a little disheartening, but first consider an example. Suppose an experimenter wishes to discover how many times the circumference of a circle can be stretched across its diameter. Of course the answer is π, but determining the exact value of π experimentally is in fact impossible. If the measuring instrument used was a meter stick, the result could only be determined as accurately as the smallest division on the meter stick, at best. Since π is irrational, it can never be measured exactly. There will always be some error.

Having obtained a result, the next question is, "How representative is the result?" In lieu of answering this question, consider another example. Suppose a sharpened pencil and a ruler are given to a group of twenty people, and they are each asked to measure the pencil and to report their results as accurately as possible. It is a safe bet that each member of the group will not report the same result. If you divide the measurements into "bins" that are one millimeter wide, i.e. an even millimeter ± a half-millimeter, and plot a graph of the number of measurements in each bin versus the bin number, in this case the even millimeter number, you would expect something like Figure 2.

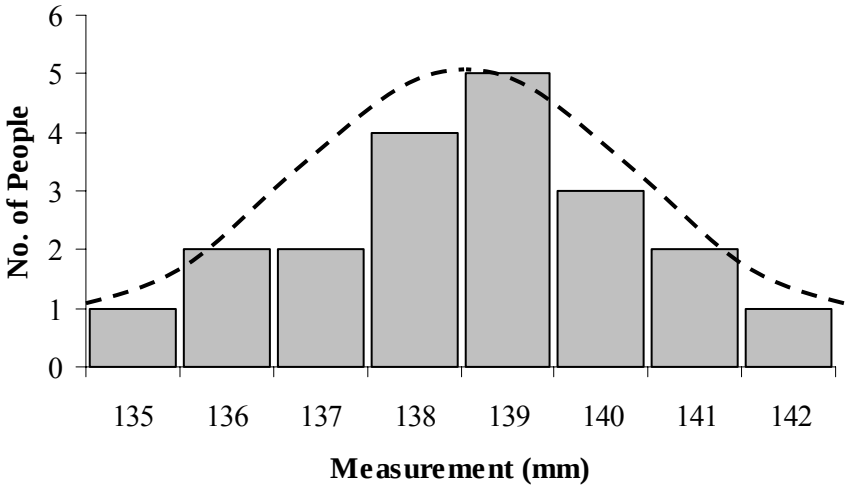


Figure 2 Histogram of Pencil Measurements

The part of the graph with the steps is called a histogram. The curve drawn in dashed is called a Gaussian, or a bell, curve and describes what is known as a *normal distribution*. It is a symmetrical curve with its line of symmetry being a vertical line that passes through its maximum. Lastly, the curve can be described mathematically by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(x-\bar{x})^2}{2\sigma^2}\right)} \tag{1}$$

This may look a bit intimidating, but what is important at this point is that the curve can be described mathematically. If the errors are truly random and there are a large number of measurements and the count of people within smaller measurement intervals are plotted, the blockiness of the steps wash out and the distribution approaches the normal distribution. In most instances, this is not a bad assumption. As the number of measurements approaches infinity, the curve maximum becomes the distribution mean, μ.

Suppose someone now asks for the length of the pencil. What is the best result? Given the twenty measurements, all different, the best result is the average, or the mean, $\bar{x}$, of all twenty measurements. If the measurements are designated $x_1, x_2, x_3, \ldots, x_n$ or simply x_i, then

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{(x_1 + x_2 + x_3 + \ldots + x_n)}{n} \tag{2}$$

where n is the number of measurements.

In the example, n = 20. So if many measurements of some characteristic have been taken, the best value for that characteristic is the mean value of all the measurements. Now, back to "How representative is the measurement?" Suppose a measuring experiment is performed twice, each time taking four measurements. The results are:

Experiment 1 – 4,5,5,6 cm Experiment 2 – 3,5,5,7 cm

A quick mental calculation will verify that both of these experiments yield an average value of 5 cm. However, in Experiment 2, the people who failed to measure the value 5 cm reported a value further from 5 cm than was the case in the first experiment. Therefore, greater confidence can be placed in the reported value of 5 cm in Experiment 1 than in Experiment 2. To relate this to the normal distribution, this corresponds to a narrower Gaussian distribution, i.e. the measurements are grouped tighter, for the first experiment than the second experiment and introduces the concept of a standard deviation. A standard deviation describes how fat or wide a Gaussian distribution is. One would like the standard deviation to describe the distance from the mean, $\bar{x}$, to a typical element, x_i, of the distribution. Thus, a distribution with a large standard deviation has its elements farther away from its mean than does a distribution with a small standard deviation. At first glance, an expression for the measure of scatter might be:

$$\text{Stan. Dev.?} = ?\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}) \tag{3}$$

This is, supposedly, the average deviation from the mean. However, this formulation will not work, since it will always equal zero as prescribed by the definition of $\bar{x}$. One might get around this problem by taking the absolute value of the deviations. This measure of scatter is known as the mean absolute deviation (MAD).

$$\text{MAD} = \frac{1}{n}\sum_{i=1}^{n}\left|x_i - \bar{x}\right| \tag{4}$$

However, MAD suffers from the undesirable features of the absolute value function, such as being non-differentiable at $x_i = \bar{x}$. Summing the squares of the deviations, dividing by *n* and taking the square root of the quotient circumvents these problems. This is called root mean square (R.M.S.) deviation and looks like:

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} \tag{5}$$

As the number of observations, *n*, approaches infinity, this is the standard deviation, σ, of the distribution and is the basic statistic used to describe the scatter of the infinite population of observations, just as μ is the distribution mean.

In an actual experiment, an infinite number of observations cannot be taken. Therefore, estimates of μ and σ are needed to describe the best value and scatter of the measurements. The sample mean, $\bar{x}$, is an unbiased estimate of the population mean, μ[1], and the sample standard deviation is an unbiased estimate of the population standard deviation, σ[1]. However, the calculation of the sample standard deviation differs slightly from the calculation of the population standard deviation. In this case the summation of the deviations squared is divided by (*n*-1) instead of *n*, since there are only *n*-1 independent observations, i.e. x_n can be determined from $x_n = n\bar{x} - \sum_{i=1}^{n-1} x_i$. If multiple samples are taken from a population having a mean, μ, and standard deviation, σ, and the mean, $\bar{x}$, is calculated for each sample, these means are normally distributed with the same mean, μ, and a standard deviation that is smaller than the standard deviation of the population by an inverse-square-root relationship. The equations for the sample standard deviation, s_x, and the associated sample standard deviation of the mean, $s_{\bar{x}}$, are:

$$\begin{aligned} s_x &= \sqrt{\frac{1}{(n-1)}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} \\ s_{\bar{x}} &= \frac{s_x}{\sqrt{n}} \end{aligned} \tag{6}$$

Notice that the sample standard deviation of the mean, $s_{\bar{x}}$, decreases as *n* increases since s_x remains approximately equal to σ, a constant for all *n*. This means that by making more measurements, we can

narrow the value of $\bar{x}$ down to a smaller interval. This is the primary motivation in repeating a measurement many times, to get closer to the true value of μ.

Having obtained an estimate of the scatter of a set of measurements, the question immediately arises as to how good these estimates are. Since not all elements of the population were measured, and since the sample was selected completely at random, there is always the possibility that the sample may not be representative of the population. One can never say with absolute certainty how far the estimates, $\bar{x}$ and *s*, are off from their true values, μ and σ, since one does not know the true values. The best that can be done is give an interval and quantify the probability that μ will fall within that interval.

Equation (1) is in normalized form such that

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(x-\bar{x})^2}{2\sigma^2}\right)} dx = 1 \tag{7}$$

This result is intuitive since the probability of a measurement being between plus and minus infinity must equal 1. The probability of the measurement from the Gaussian parent population falling within a specified range, for example $\pm\Delta x$, will then be:

$$\text{Prob}(\Delta x) = \int_{\mu-\Delta x}^{\mu+\Delta x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)} dx \tag{8}$$

When the specified ranges are $\pm 1\sigma$ and $\pm 2\sigma$ the probabilities of a measurement from the Gaussian parent population falling within these ranges are approximately 0.68 and 0.95 respectively.

The integral in Equation 8 cannot be evaluated in closed form, and if its values were tabulated, a separate table would be needed for an infinite number of (μ,σ) pairs. Therefore, normalizing this integral by defining the normalized deviation from the mean value as $\tau = \frac{x-\mu}{\sigma}$, requires only a single table, Appendix I. In this form the integral is:

$$\text{Prob}(\tau_1) = \frac{1}{\sqrt{2\pi}} \int_{-\tau_1}^{\tau_1} e^{-\tau^2/2}\, d\tau \tag{9}$$

Where:

$$\tau_1 = \frac{\Delta x}{\sigma} = \frac{x_i - \mu}{\sigma} \tag{10}$$

From the table, Appendix I, generated from Equation 9 and using Equation 10, a confidence interval can be constructed such that one is confident to a stated percentage that the measurement, x_i, is within a determined tolerance of the population mean. In other words, the stated percentage of the Gaussian population lies with the pre-determined interval about the mean. For example a 95% confidence interval is:

$$\text{Prob}\left(-1.96 \le \frac{x_i - \mu}{\sigma} \le 1.96\right) = 0.95 \tag{11}$$

Rearranging to isolate x_i or μ:

$$\begin{aligned} &\text{Prob}(\mu - 1.96\sigma \le x_i \le \mu + 1.96\sigma) = 0.95 \\ &\text{Prob}(x_i - 1.96\sigma \le \mu \le x_i + 1.96\sigma) = 0.95 \end{aligned} \tag{12}$$

This confidence interval states that 95% of the population of a Gaussian distribution lies with $\pm 1.96\sigma$ of the population mean, so one can be 95% confident that a single measurement will fall within that interval.

However, the above confidence interval assumes a parent population with an infinite number of observations. In any practical situation only a sample from the population will be available, and therefore, $\bar{x}$ and s_x will be calculated instead of μ and σ. Recalling that $\bar{x}$'s are normally distributed, an analog to Equation (11) can be written using the t distribution, Appendix II:

$$\begin{aligned} &\text{Prob}\left(-t \le \frac{x_i - \mu}{s_x} \le t\right) = 0.95 \text{ or} \\ &\text{Prob}\left(-t \le \frac{\bar{x} - \mu}{s_x / \sqrt{n}} \le t\right) = 0.95 \end{aligned} \tag{13}$$

The variables $\frac{x_i - \mu}{s_x}$ and $\frac{\bar{x} - \mu}{s_x / \sqrt{n}}$ are not normally distributed, but they follow the *t* distribution with *n*-1 degrees of freedom, ν. The values of *t* that satisfy Equations (13) are dependent on the sample size, *n*. As *n* approaches infinity, the values of *t* approach those for a Gaussian distribution, i.e. for a 95% confidence interval the *t* value approaches 1.96 for large *n*.

Finally we isolate μ in Equations (13):

$$\begin{aligned} &\text{Prob}(x_i - ts_x \le \mu \le x_i + ts_x) = 0.95 \text{ or} \\ &\text{Prob}\left(\bar{x} - t\,s_x/\sqrt{n} \le \mu \le \bar{x} + t\,s_x/\sqrt{n}\right) = 0.95 \end{aligned} \tag{14}$$

and define random uncertainties P_x and $P_{\bar{x}}$ as:

$$P_X = ts_X \text{ and}$$
$$P_{\bar{X}} = t\left(\frac{s_X}{\sqrt{n}}\right) \tag{15}$$

4. LOWER LIMIT ON MEASUREMENT ERRORS

Equation 6 for the sample standard deviation of the mean, $s_{\bar{X}}$, implies that one can determine any quantity to arbitrary precision simply by repeating the measurement enough times, since

$$\lim_{n\to\infty} \sigma_{\bar{X}} = \lim_{n\to\infty} \frac{s_X}{\sqrt{n}} = 0 \tag{16}$$

However, this assumes that each element in the sample x_i, $i = 1...n$ was measured with " arbitrary precision", i.e. that each x_i had an infinite number of significant digits. This is not very realistic, because each measurement with an instrument can have only a finite number of significant digits.

To illustrate the point, suppose a given mass is measured with a digital balance and suppose the balance reads 152.4 grams. Probably the same reading will be obtained no matter how many times the object is weighed. Should one conclude that the object is exactly 152.4000... grams? The answer is, "No", and the explanation is that the balance rounds the actual mass of the object to 4 digits. Therefore, all that the digital reading is indicating, at best, is that the actual mass was anywhere between 152.35 grams and 152.45 grams. The measurement at least deviates by ±0.05 grams. Therefore, the standard deviation of the mean for a digital instrument is ½ the smallest measuring unit (smu) (or *least count* as some people call it), and digital measurements are uncertain by $\frac{\tau_{\%}}{2}$ smu.

The situation is not very different with analog gauges. The smallest division on a meter stick is usually 1 mm, and although one can roughly estimate to a tenth of a mm, that is as far as one can go. Assuming that any estimation within that smallest division is equally likely, i.e. the distribution of estimates is a uniform distribution, the standard deviation is the standard deviation of a uniform distribution with an interval bounded by parameters α and β, where $(\beta - \alpha)$ is the smallest measuring unit, and since the number of samples for any particular reading is one, $\sigma = \sigma_{\bar{X}}$. Multiplying $\sigma_{\bar{X}}$ by $\tau_{\%}$ from the Gaussian probability table in Appendix I gives the minimum uncertainty, $U_{\bar{X}}$, for any particular measurement to a specified confidence level.

Since each measurement in a sample taken with analog measurements is uncertain by $\frac{\tau_{\%}}{\sqrt{12}}$ of the smallest division and each measurement taken with a digital instrument is uncertain by $\frac{\tau_{\%}}{2}$ of the smallest measuring unit, $\bar{x}$ itself cannot be more precise than these. Therefore $U_{\bar{X}}$ is bounded from below by $\frac{\tau_{\%}}{\sqrt{12}}$smu for analog measuring devices and $\frac{\tau_{\%}}{2}$ sum for digital measuring devices:

$$U_{\bar{X}} = \text{MAXIMUM}\left(\frac{t_{\%} s_X}{\sqrt{n}}, \frac{\tau_{\%}}{\sqrt{12}} \text{smallest analog division} \ \textbf{or} \ \frac{\tau_{\%}}{2} \text{smallest digital division} \right) \quad (17)$$

5. SIGNIFICANT FIGURES

While some statistical calculations, particularly those involving the differences of large numbers, require carrying an inordinate number of significant digits, it is never appropriate to report calculated results with more significant figures than any of the important numerical values that were utilized in the calculation. An example where a large number of significant digits must be carried is the calculation of differences between summations in analysis of variance. As a general rule, unless there is a compelling reason such as the example cited above, carry one or two extra significant digits in intermediate calculations and report the final result with no more significant digits that exist in the least of the important parameters used in the calculation.

6. STATISTICAL REJECTION OF OUTLIERS FROM A SAMPLE (Adapted from Coleman and Steel[2])

When a sample of *n* measurements is examined, some measurements may appear "off". If verifiable sources of error can be identified for these outliers or wild points, they should be either corrected, if possible, or discarded. Usually, this is not the case, and a statistical criterion must be used to find points that can be rejected. There is no other justifiable way to "throw away" data points.

A widely accepted method is Chauvenet's criterion. The criterion states that points should be retained if they fall within a band around the mean that corresponds to a probability of $1 - \frac{1}{2n}$. This criterion uses the Gaussian probabilities in Appendix I such that:

$$\begin{aligned} 1 - \frac{1}{2n} &= \text{Prob}(\tau) \ \text{where} \\ \tau &= \frac{x_i - \bar{x}}{s_x} = \frac{\Delta x_{\max}}{s_x} \end{aligned} \quad (16)$$

For example if a sample of 8 measurements are taken, $1 - \frac{1}{2n} = 1 - \frac{1}{16} = 0.9375$. From Appendix I, τ is approximately 1.864, obtained by interpolating between the table τ values, 1.86 and 1.88. The maximum deviation of points to be retained is then 1.864 times the sample standard deviation. Points that fall outside this deviation are rejected and a new sample mean and standard deviation are calculated. **The Chauvenet's criterion should never be applied more than once.**

7. SYSTEMATIC ERRORS

Systematic errors, β, often referred to as determinant or constant errors, arise from variables in an

experiment that are ignored. Frequently acceleration of gravity is measured by timing how long it takes a heavy object to drop a known distance and then using those values in the kinematic equations. In doing this, the effects of air resistance are ignored. This is all right so long as the object does not respond to air resistance very much. The error is still there, but because it is small, it is neglected. However, had the experiment been performed with a feather, the effect of the air on the object could not be ignored. The only way to totally ignore the effect of the air is to perform the experiment in an air-free environment.

Often systematic errors are due to inaccuracies in the measuring devices used. If a ruler has one end worn away, it will always give a length longer than the true length. Another form of systematic error is personal bias, which can only be minimized by an objective attitude. In general, the mode taken to minimize systematic error is to minimize the effects of the variables we are ignoring.

Estimating systematic errors is difficult. One approach is to assume that any one instance of systematic error, B, is drawn from a population of possible systematic errors, and that population of possible systematic error has a Gaussian distribution. Then the interval defined by $\pm B = \pm t s_B$ gives an estimate of the systematic uncertainty. For the 95% confidence level $t \approx 2$, and this value is commonly used such that $\pm B = \pm 2 s_B$. The manufacturers of instruments sometime supply estimates of s_B, but often the experimenter must compare the instruments to other standards or make educated guesses. Where experimental influences are ignored, such as air resistance in the above example, the experimenter should model the phenomenon producing the influence to show that the effect is indeed negligible.

7. REFERENCES

[1] Johnson, Richard A., *Miller & Freund's Probability and Statistics for Engineers*, Sixth Edition, Prentice Hall, New Jersey, 2000.

[2] Coleman, Hugh W. and Steele, W. Glenn Jr., *Experimentation and Uncertainty Analysis for Engineers*, Second Edition, John Wiley & Sons, Inc., New York, 1999.

[3] Steel, Robert G. D. and Torrie, James H., *Principles and Procedures of Statistics*, McGraw-Hill Book Company, New York, 1960.

Appendix I – Tabulated values of the τ Probabilities[1]

Tabulation of Two-Tailed Gaussian Probabilities

τ	Prob(τ)	τ	Prob(τ)	τ	Prob(τ)	τ	Prob(τ)
0.00	0.0000	1.00	0.6827	2.00	0.9545	3.00	0.9973002
0.02	0.0160	1.02	0.6923	2.02	0.9566	3.05	0.9977115
0.04	0.0319	1.04	0.7017	2.04	0.9586	3.10	0.9980647
0.06	0.0478	1.06	0.7109	2.06	0.9606	3.15	0.9983672
0.08	0.0638	1.08	0.7199	2.08	0.9625	3.20	0.9986257
0.10	0.0797	1.10	0.7287	2.10	0.9643	3.25	0.9988459
0.12	0.0955	1.12	0.7373	2.12	0.9660	3.30	0.9990331
0.14	0.1113	1.14	0.7457	2.14	0.9676	3.35	0.9991918
0.16	0.1271	1.16	0.7540	2.16	0.9692	3.40	0.9993261
0.18	0.1428	1.18	0.7620	2.18	0.9707	3.45	0.9994394
0.20	0.1585	1.20	0.7699	2.20	0.9722	3.50	0.9995347
0.22	0.1741	1.22	0.7775	2.22	0.9736	3.55	0.9996147
0.24	0.1897	1.24	0.7850	2.24	0.9749	3.60	0.9996817
0.26	0.2051	1.26	0.7923	2.26	0.9762	3.65	0.9997377
0.28	0.2205	1.28	0.7995	2.28	0.9774	3.70	0.9997843
0.30	0.2358	1.30	0.8064	2.30	0.9786	3.75	0.9998231
0.32	0.2510	1.32	0.8132	2.32	0.9797	3.80	0.9998552
0.34	0.2661	1.34	0.8198	2.34	0.9807	3.85	0.9998818
0.36	0.2812	1.36	0.8262	2.36	0.9817	3.90	0.9999037
0.38	0.2961	1.38	0.8324	2.38	0.9827	3.95	0.9999218
0.40	0.3108	1.40	0.8385	2.40	0.9836	4.00	0.9999366
0.42	0.3255	1.42	0.8444	2.42	0.9845	4.05	0.9999487
0.44	0.3401	1.44	0.8501	2.44	0.9853	4.10	0.9999586
0.46	0.3545	1.46	0.8557	2.46	0.9861	4.15	0.9999667
0.48	0.3688	1.48	0.8611	2.48	0.9869	4.20	0.9999732
0.50	0.3829	1.50	0.8664	2.50	0.9876	4.25	0.9999786
0.52	0.3969	1.52	0.8715	2.52	0.9883	4.30	0.9999829
0.54	0.4108	1.54	0.8764	2.54	0.9889	4.35	0.9999863
0.56	0.4245	1.56	0.8812	2.56	0.9895	4.40	0.9999891
0.58	0.4381	1.58	0.8859	2.58	0.9901	4.45	0.9999911
0.60	0.4515	1.60	0.8904	2.60	0.9907	4.50	0.9999931
0.62	0.4647	1.62	0.8948	2.62	0.9912	4.55	0.9999946
0.64	0.4778	1.64	0.8990	2.64	0.9917	4.60	0.9999957
0.66	0.4907	1.66	0.9031	2.66	0.9922	4.65	0.9999966
0.68	0.5035	1.68	0.9070	2.68	0.9926	4.70	0.9999973
0.70	0.5161	1.70	0.9109	2.70	0.9931	4.75	0.9999979
0.72	0.5285	1.72	0.9146	2.72	0.9935	4.80	0.9999984
0.74	0.5407	1.74	0.9181	2.74	0.9939	4.85	0.9999987
0.76	0.5527	1.76	0.9216	2.76	0.9942	4.90	0.9999990
0.78	0.5646	1.78	0.9249	2.78	0.9946	4.95	0.9999992
0.80	0.5763	1.80	0.9281	2.80	0.9949	5.00	0.9999994
0.82	0.5878	1.82	0.9312	2.82	0.9952		
0.84	0.5991	1.84	0.9342	2.84	0.9955		
0.86	0.6102	1.86	0.9371	2.86	0.9958		
0.88	0.6211	1.88	0.9399	2.88	0.9960		
0.90	0.6319	1.90	0.9426	2.90	0.9963		
0.92	0.6424	1.92	0.9451	2.92	0.9965		
0.94	0.6528	1.94	0.9476	2.94	0.9967		
0.96	0.6629	1.96	0.9500	2.96	0.9969		
0.98	0.6729	1.98	0.9523	2.98	0.9971		

Appendix II – Tabulated values of the *t* distribution[3]

Values of *t*

	Probability of a larger value of *t*, sign ignored - two tailed								
df	0.5	0.4	0.3	0.2	0.1	0.05	0.02	0.01	0.001
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	636.619
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	31.598
3	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	12.941
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	6.859
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	5.405
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.767
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.551
60	0.679	0.848	1.046	1.296	1.671	2.000	2.390	2.660	3.460
120	0.677	0.845	1.041	1.289	1.658	1.980	2.358	2.617	3.373
inf.	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.291
df	0.25	0.2	0.15	0.1	0.05	0.025	0.01	0.005	0.0005
	Probability of a larger value of *t*, sign considered - one tailed								