Eigenvalue Inclusion Regions

It is not uncommon to work with large matrices (i.e. with over 10K rows/columns). Finding e vals exactly is impossible in this setting. But often we don't need the e vals exactly. We just need to know their rough location.

Eg. Say we are solving

\[ \dot{X} = Ax \quad \text{where} \quad A \text{ is d-ble} \]

Say \( P^{-1}AP = D = \text{diag}(d_1, \ldots, d_n) \)

Solution will be

\[ X = P \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix} X_0 \]

To determine long term behavior of \( X \) need to look at \( \lim_{t \to \infty} e^{at} \)

Say \( X = a + bi \). \( \lim_{t \to \infty} e^{(a+bi)t} \)

= \( \lim_{t \to \infty} e^{at} \cos(bt + \phi) \)

= \( \lim_{t \to \infty} e^{at} \) and does not exist if \( a = 0 \)
Defn: \( A_{n \times n} \) is **stable** if each of its 
eigenvalues has \( \text{Re}\{\lambda\} < 0 \).

Another example:

\( x(0) \) \( n \times 1 \) vector

\( A \) \( n \times n \) matrix

\( x(k+1) = Ax(k) \) for \( k \geq 0 \).

Say \( AP = D, D \) diagonal \( (d_{ii} > 0) \).

Then \( x(k+1) = PD^kP^{-1}x_0 \).

So

\[
\lim_{k \to \infty} x_k \text{ depends on } \lim_{t \to 0} d_+^t = \begin{cases} 0 & \text{if } d_i > 1 \\
1 & \text{otherwise}
\end{cases}
\]

Defn: \( A \) is ** Perron-stable** if each eigenvalue of \( A \) is less than 1.
Desire simple ways to find regions that include all e-values of $A$.

**Gershgorin's Thm**

Let $A = [a_{jk}]$ be an $n \times n$ complex matrix. Let $R_j = \sum_{k \neq j} |a_{jk}|$ be the $j$th punctured absolute row sum.

Let

$$G = \left\{ \mathbf{z} \in \mathbb{C} : |z - a_{jj}| \leq R_j \right\}.$$

Then each e-value of $A$ is in $\bigcup_{j=1}^n G_j$.

**Example**

$$A = \begin{bmatrix} 4 & i & 2 \\ 0 & 3i & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

$R_1 = |1| + 2 = 3$

$R_2 = 0 + 1 = 1$

$R_3 = 1$
So $A$ is stable.

Proof of Geršgorin's

Suppose $x$ is an $e$-value of $A$.

Let $x$ be a corresponding e-vector.

Among all $x_k$'s choose $j$ s.t. $|x_j|$ is largest.

Now $A x = \lambda x$ says

$$\sum_{k=1}^{n} a_{jk} x_k = \lambda x_j$$

So

$$\sum_{k=1}^{n} a_{jk} x_k = (\lambda - a_{jj}) x_j$$

Thus

$$\|x_j\|^2 \geq |\sum_{k=1}^{n} a_{jk} x_k| = |\lambda - a_{jj}| |x_j|$$

\[\Delta = \text{diag}(A) \]
Thus, $|a_{ij}| \leq R_j$, as desired.

**Defn** $A_{n \times n} = [a_{ij}] \text{ is diagonally dominant if}$

$|a_{jj}| > \sum_{i \neq j} |a_{ij}|$ for each $j$.

**Eg**

$$
\begin{bmatrix}
4 & 1 & 1 \\
0 & 2 & 1 \\
1 & 1 & 3
\end{bmatrix}
$$

is $d$-dom.

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**Lavé-Deshphnques Thm**

If $A$ is diagonally dominant, then $A$ is invertible.

**Proof**. Spce $A$ is diag. dominant.

By Gersgorin's thm., 0 is not an $\lambda$-value of $A$. This means the nullspace of $A = 0$. So $A$ is invertible.