1. Let $L$ be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by

$$L \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + 2z \\ 2x + y - z \end{bmatrix}.$$ 

(a) Find $M^L_{\alpha,\beta}$ where

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$ 

(b) Find bases $\gamma$ and $\delta$ so that

$$M^L_{\gamma,\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$  

(Hint: you’ll need to find a nonzero vector $v$ so that $L(v) = \vec{0}$.)

2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}.$$ 

We wish to determine all 4 by 3 matrices $X$ for which $AXA = A$. Observe that

$$PAQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $P$ and $Q$ are the invertible matrices

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ 

(a) Show how you can use this to convert the problem of finding all $X$ so that $AXA = A$ to finding all 4 by 3 matrices $Y$ so that $BYB = B$, where $B = PAQ$.

(b) Find all $Y$ so that $BYB = B$.

(c) Use your answer in (b) to find all $X$ so that $AXA = A$.

(d) Give 2 explicit $X$ with $AXA = A$. 

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Matrix Theory
Homework # 5
Due Wednesday, October 27
3. Let \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) by
\[
L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y + z \\ x + 2y + z \\ x + y + 2x \end{bmatrix}.
\]
(a) Find \( M_{ST,ST}^L \).
(b) Find \( M_{\alpha,\alpha}^L \) where
\[
\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.
\]
(c) How are your answers to (a) and (b) related?
(d) Using (b) describe what \( L \) does to \( \mathbb{R}^3 \) geometrically.

4. Let
\[
A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.
\]
(a) Find an invertible matrix \( P \) so that \( P^{-1}AP = \text{diag}(4, 1, 1) \). (Hint: your answer to 3 (b) should help).
(b) We wish to determine all matrices \( B \) so that \( AB = BA \). Show that this can be reduced to determining all matrices \( C \) so that \( DC = CD \), where \( D = \text{diag}(4, 1, 1) \).
(c) Determine all \( C \) so that \( DC = CD \).
(d) Determine all \( B \) so that \( AB = BA \) (your answer can be in terms of \( C \)'s and \( P \)).

5. Let \( A \) and \( B \) be \( n \) by \( n \) matrices, and let \( P \) be an invertible matrix with \( B = P^{-1}AP \).
(a) Show that if \((\lambda, x)\) is an eigenpair of \( A \), then \( \lambda \) is an eigenvalue of \( B \). What is a corresponding eigenvector?
(b) Show that if \((\mu, y)\) is an eigenpair of \( B \), then \( \mu \) is an eigenvalue of \( A \). What is a corresponding eigenvector?
(c) Based on (a) and (b) what can you say about eigenvalues of similar matrices?

6. Let \( A \) be an \( n \) by \( n \) matrix with eigenpairs \((2, u), (2, v)\) and \((3, w)\).
(a) Show that \( u \) is an eigenvector of the matrix \( A^2 \), and find the corresponding eigenvalue. Let \( p(x) \) be a polynomial. Show that \((p(2), u)\) is an eigenpair of \( A \).
(b) Show that either \( u + v = 0 \) or \( u + v \) is an eigenvector of \( A \), and find the corresponding eigenvalue.

(c) Show that \( u + w \) is not an eigenvector of \( A \).

(d) Show that if \( A \) is invertible, then both \( 1/2 \) and \( 1/3 \) are eigenvalues. (Hint: find corresponding eigenvectors.)

7. Consider an ecosystem in which there are roadrunners and coyotes. We let \( r(t) \) and \( c(t) \) denote the populations of roadrunners and coyotes at time \( t \). The roadrunners naturally reproduce at a rate of 1.4 roadrunners/year. However, the coyote population affects the road-runners by eating some of the young. In fact each coyote eats 1.2 roadrunners per year. Thus the net birth rate of road runners is given by \( \dot{r} = 1.4r - 1.2c \). The net birth rate of coyotes is given by \( \dot{c} = .8r - 1.4c \). The .8r reflects the fact that more roadrunners around helps feed the coyotes which results in more coyotes. The \(-1.4c\) reflects the fact that coyotes compete with each other—so more coyotes around has a negative impact on birth rate. Our goal is to determine a formula for \( r(t) \) and \( c(t) \).

Note that we can write all this information as:

\[
\begin{bmatrix}
\dot{r} \\
\dot{c}
\end{bmatrix} =
\begin{bmatrix}
1.4 & -1.2 \\
.8 & -1.4
\end{bmatrix}
\begin{bmatrix}
r \\
c
\end{bmatrix},
\]

(1)

(a) Show that

\[
\begin{bmatrix}
3 \\
1
\end{bmatrix}
\text{ and } 
\begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

are eigenvectors of the coefficient matrix (let’s call it \( A \)) in (1).

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) so that \( P^{-1}AP = D \).

(c) Show that solving (1) can be reduced to solving

\( \dot{y} = Dy \),

where \( y = P^{-1}x \).

(d) Solve \( \dot{y} = Dy \).

(e) Solve (1), using \( r(0) = 100 \) and \( c(0) = 10 \).

(f) Based on your answer to (d), describe what the roadrunner population looks like in the long run. (That is, in the long run do the roadrunners die out, or does the population become steady, or does it grow exponentially?)

(g) Based on your answer to (d), what will \( r(t)/c(t) \) approach in the long run?