



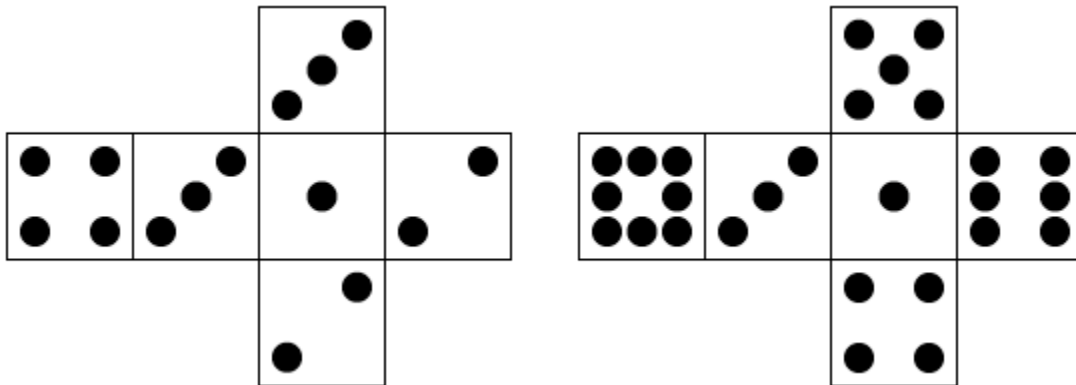
## Sicherman Dice

When we roll two ordinary dice and add the outcomes, we get the numbers between two and twelve with different probabilities: The chance of getting seven is  $1/6$ , whereas the chance of getting two is only  $1/36$ .

In one of his *Mathematical Games* columns in *Scientific American* from 1978, later reprinted in *Penrose Tiles to Trapdoor Ciphers*, Martin Gardner describes the following problem, originally asked and answered by George Sicherman of Buffalo, NY:

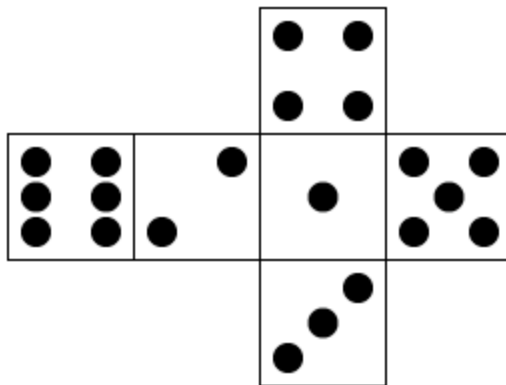
Is it possible to number the sides of two dice with positive integers in some other way, and still have the sums give the numbers 2,...,12 with the same probabilities as with ordinary dice?

It turns out that there is in fact one other way of doing it, by numbering the sides of one die with the numbers 1, 2, 2, 3, 3 and 4, and those of the other with 1, 3, 4, 5, 6 and 8:

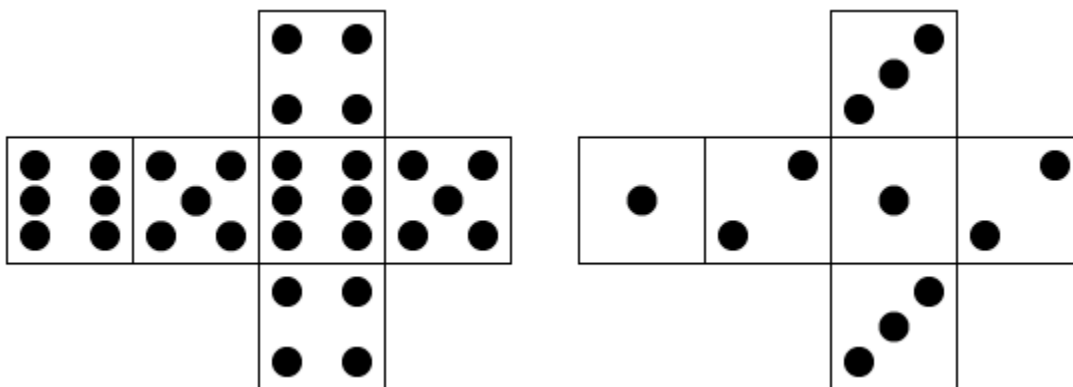


Of course, this can be verified simply by making a table and checking that there is one way of getting 2, two ways of getting 3, etc. However, there is a more algebraic approach, that will also show that this is in fact the *only* alternative numbering:

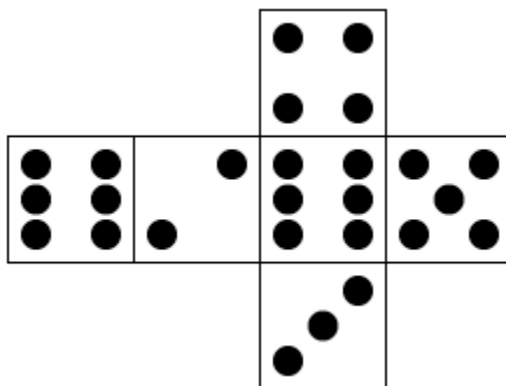
To a die we associate a *generating polynomial* by taking a term  $x^d$  for each side of the die with value  $d$ . Thus, for an ordinary die



the generating polynomial is  $x^6+x^5+x^4+x^3+x^2+x$ , whereas the generating polynomials for the two Sicherman dice above are  $x^4+2x^3+2x^2+x$  and  $x^8+x^6+x^5+x^4+x^3+x$ , respectively. Other examples: In some stores, it is possible to buy a pair of false dice, marked like this:



Since normally only three sides of a die are visible at a time, they will look like ordinary dice, while increasing your chances of rolling 7 from 1/6 to 1/3. Their generating polynomials are, respectively,  $2x^6+2x^5+2x^4$  and  $2x^3+2x^2+2x$ . Similarly, the generating polynomial for the false die



is  $2x^6+x^5+x^4+x^3+x^2$ .

All this works for any kind of die (with non-negative numbering), regardless of the number of sides. Gaming dice with four, ten, twelve, etc., sides have generating polynomials as well.

In order for a polynomial  $f(x)$  to be the generating polynomial for *some* die, it is necessary and sufficient that all the coefficients in  $f(x)$  are non-negative integers. In that case, the number of sides on the die equals  $f(1)$ .

Now, if we take two dice, with generating polynomials  $f(x)$  and  $g(x)$ , and roll them together, adding up the outcomes, they can be considered as a single "compound" die, and this die will have generating polynomial  $f(x)g(x)$ : If die #1 rolls  $a$  and die #2 rolls  $b$ , this corresponds to a degree- $a$  term in  $f(x)$  and a degree- $b$  term in  $g(x)$ , which in turn produces a degree- $(a+b)$  term in  $f(x)g(x)$ .

For example, the generating polynomial for rolling two ordinary dice is

$$(x^6+x^5+x^4+x^3+x^2+x)^2 = x^{12}+2x^{11}+3x^{10}+4x^9+5x^8+6x^7+5x^6+4x^5+3x^4+2x^3+x^2.$$

We can now reformulate Sicherman's question as follows: Is it possible to write the above polynomial as a product  $f(x)g(x)$  of two polynomials  $f(x)$  and  $g(x)$  with non-negative integer coefficients and  $f(1)=g(1)=6$ ?

The irreducible factorisation of the polynomial is

$$(x+1)^2(x^2+x+1)^2(x^2-x+1)^2,$$

and if  $f(x)$  and  $g(x)$  are to have a value of 6 for  $x=1$ , they must both have a factor  $x+1$  and a factor  $x^2+x+1$ . This leaves only the two factors  $x^2-x+1$  to be distributed. If we put one on each polynomial, we get two ordinary dice, and if we put both on the same polynomial, we get the two Sicherman dice. This establishes both existence and uniqueness of the alternative numbering: It can be done in *exactly* one way.

It is possible to consider more than two dice this way, but it does not give any new results: If we want to number  $n$  dice with positive integers, such that they give the same distribution of sums as  $n$  ordinary dice, the only possibility is to replace pairs of ordinary dice with pairs of Sicherman dice: We now need to distribute factors on  $n$  polynomials, and it is still necessary that each polynomial gets factors  $x+1$  and  $x^2+x+1$ , leaving us with  $n$  copies of  $x^2-x+1$  to distribute. But note the following about the polynomials given above: If there is no factor  $x^2-x+1$ , the quadratic term has coefficient 2; if there is one factor, the quadratic term has coefficient 1; and if there are two factors, the quadratic term has coefficient 0. It is quite easy to see by induction that the quadratic term will have coefficient  $2-d$ , if there are  $d$  factors  $x^2-x+1$ . Consequently, we cannot have more than two, leaving us with only ordinary dice and Sicherman dice, no matter what  $n$  is.

## References

J. A. Gallian & D. J. Rusin, *Cyclotomic polynomials and nonstandard dice*, *Discrete Mathematics* **27** (1979), pp. 245-259.  
M. Gardner, *Penrose Tiles to Trapdoor Ciphers*, Spectrum, Mathematical Association of America.

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