Review

- Multiplication by a transposition changes the parity.
  if f is odd, τf is even;
  if f is even τf is odd.
- Every permutation is a product of transpositions
  A product of an odd number of transpositions is an odd permutation
  A product of an even number of transpositions is an even permutation
- For permutations we have:

$\circ$	Even	Odd
Even	Even	Odd
Odd	Odd	Even

Practice

Let $f = (1, 2, 4, 5, 6)(7, 8, 9)$

- Write f as a product of transpositions
- Is f an even permutation or an odd permutation?
- Is f^2 an even or odd permutation?
- Is $(1, 2)f(1, 2, 3)$ an even or odd permutation?
- Is there an even permutation such that $f(1, 2, 3) = (1, 3)$? Why or why not?

Train track problem

You have an oval train track; 11 cars in order 1,2,3,. . . , 11; a round-about that allows you to reverse the order of a section of 4 trains; and you can move the trains around the oval.

Can you ever get the trains to be in the order $1, 2, 4, 3, 5, 6, 7, 9, 10, 8, 11$?

Valid basic operations correspond to $f = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$ which is an even permutation, and $g = (1, 4)(2, 3)$ which is an even permutation. So any product of f's and g will be even.

The final configuration you want corresponds to the permutation $(3, 4)(8, 10, 9)$ which is an odd permutation.

So it is impossible!

Perfect Shuffles of a deck of 16 cards: Homework

1. Find the permutation, ℓ corresponding to a left shuffle.
2. Find the permutation r corresponding to a right shuffle.
3. Find the permutation corresponding to a left shuffle followed by a right shuffle by "physically doing the shuffle", and then by computing $r \circ \ell$.
4. What would the original deck look like after you did 5 left shuffles?
5. What would the original deck look like after you did a left, right, left, right, left, right, left, right shuffle?
6. What kind of permutation is ℓ?
7. What kind of permutation if r?
8. Could I start with the original deck, do a bunch of left or right shuffles in some order, and end up at with the cards in the order $2, 3, 4, 5, 1, 7, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16$? Why or why not?

15 puzzle: Homework

- Description (http://www.cut-the-knot.org/pythagoras/fifteen.shtml)
- Every basic move corresponds to a transposition
- Every sequence of moves that starts and ends with a blank in the lower right involves an even number of moves
- Every sequence of moves corresponds to an even permutation. Why? Why is it a product of an even number of transpositions?
- Upshot: No odd permutation of the cells 1-15 can be put into the correct order.

A **permutation group** is a subset G of permutations s.t.
(a) id $\in G$
(b) G is closed under composition
(c) G is closed under inverses.

Examples:

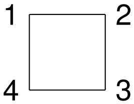
All the permutations of $\{1, 2, \ldots, n\}$. Denoted by S_n and called the symmetric group

All the even permutations of $\{1, 2, \ldots, n\}$. Denote by A_n and called the Alternating group.

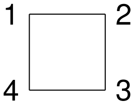
The rigid motions of a figure give permutation groups of vertices, edges, faces

The Rubik's cube gives a permutation group of the cells of the group

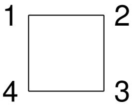
Rigid Motions of Square



R_0 corresponds to id



Rigid Motions of Square

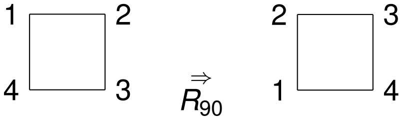


Should be 8 rigid motions.

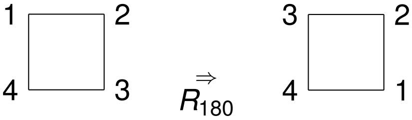
4 choices for bottom edge

2 choices for orienting edge.

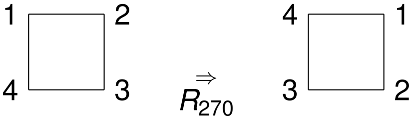
$8 = 4 \cdot 2$



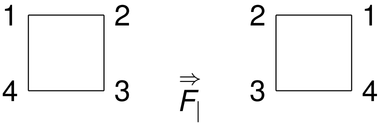
R_{90} gives permutation $(1, 4, 3, 2)$



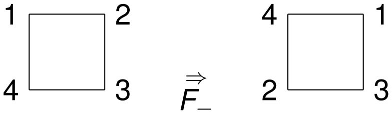
R_{180} gives permutation $(1,3)(2,4)$



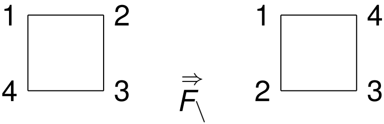
R_{270} gives permutation $(1, 2, 3, 4)$



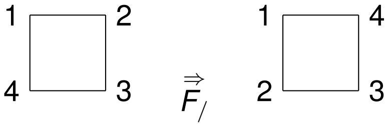
$F_|$ gives permutation $(1,2)(3,4)$



F_- gives permutation $(1, 4)(2, 3)$



$F_\backslash$ gives permutation $(2, 4)$



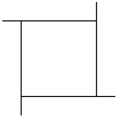
$F_/$ gives permutation $(1,3)$

Practice

- What's $F_{/}F_{\backslash}$
- What's the inverse of R_{90}? What's the inverse of $F_{|}$?
- What can you say about the product of two flips?

Rigid Motions of other objects

- A square-pinwheel



- Regular *n*-gon
- Cube
- Mattress
- Tetrahedron
- Etc

Why permutations?

- Easy family of functions to study composition, 1-1, onto, inverse, etc with.
- Easy to compute with.
- Exhibits periodic behavior–even before sin's and cos's.
- Illustrates how complex functions can be built up from simpler functions.
- Illustrates how algebra is a tool to model real-life (e.g. perfect shuffles).
- Bridge between concrete and abstract.

A **group** is an algebraic object consisting of a set G and a binary operation $\diamond$ on G such that

- $\diamond$ is associative
- There is an identity for G, $\diamond$
- Each element of G has an inverse

Examples of groups

- Any permutation group (e.g. S_n, A_n, rigid motions of an object)
- $\mathbb{Z}$ under addition
- $\mathbb{R}$ under addition
- Positive reals under multiplication
- The subsets of X under symmetric difference
- Klein-4

	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

Multiplication and Addition are the same!

Lots of different groups are disguised versions of each other.

The positive reals under multiplication is just a disguised version of the reals under addition.

Why? Because there is a dictionary from the positive reals to the reals that translates any product equation into any addition equation.

Map $f : \mathbb{R}^+ \to R$ by $f(x) = \log(x)$.

f is a good dictionary, because it is $1 - 1$ and onto.

Also the statement $z = x \cdot y$ gets translated to the true statement $\log(z) = \log(z) + \log(y)$.

So (via the logarithm) any multiplication problem becomes an addition problem.

Algebra makes life simpler!

More than one operation

Groups only have one operation.

Familiar objects like the integers, rationals, reals that have both an addition and a multiplication. Often times it is useful to have more than one operation at a time.

Let S be a set and $\heartsuit$ and $\diamond$ be binary operations on S.
We say that $\heartsuit$ is distributive over $\diamond$ on S provided for each $x, y, z \in S$ we have

$$x\heartsuit(y \diamond z) = (x\heartsuit y) \diamond (x\heartsuit z)$$

- Multiplication over addition on $\mathbb{R}$?

  Addition over multiplication on $\mathbb{R}$?
- Union over intersection on subsets of a given set?
- Maximum over addition on $\mathbb{R}$?
- Intersection over union on subsets of a given set?