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Titles & Abstracts

Minimum rank and related graph parameters

Shaun M. Fallat (University of Regina)

Given a graph $G = (V, E)$, the minimum rank of G is defined as the smallest possible rank over all real symmetric matrices $A = [a_{ij}]$ such that whenever $i \neq j$, $a_{ij} \neq 0$ if and only if $\{i, j\} \in E$. The basic objective of this lecture is to provide an overview on the relationships that exist between the minimum rank of G and the more standard graph theoretic parameters, including: minimum degree, diameter, girth, clique number, and others. As an example, I will discuss the graphs G for which the minimum rank coincides with the diameter of G.

The minimum rank problem for rectangular grids

Wayne Barrett*, H. Tracy Hall, and Hein van der Holst

The minimum rank problem for a given field F and given simple, undirected graph G is to determine the minimum rank $\mathrm{mr}(F, G)$ (or maximum nullity $M(F, G)$) over all F-valued symmetric matrices whose off-diagonal nonzero pattern corresponds to G. We also make use of a related graph parameter $\xi(G)$, which is the maximum nullity over all real symmetric matrices with graph G that also satisfy a non-degeneracy condition called the strong Arnol'd property. Given a vertex v of a graph G, the rank spread over F of G at v is

$$r_v(F, G) = \mathrm{mr}(F, G) - \mathrm{mr}(F, G - v).$$

The $m \times n$ rectangular grid is the Cartesian product of P_m and P_n, where P_m is the path of m vertices, and is denoted by $P_m \square P_n$. It is known from the AIM 2006 Workshop on spectra of families of matrices described by graphs, digraphs, and sign patterns, that $M(\mathbb{R}, P_m \square P_n) = \min\{m, n\}$.

Our first results are that for a square grid we have

1. $M(F, P_n \square P_n) = n$ for every field F.
2. $r_v(F, P_n \square P_n) = 0$ for every vertex v of $P_n \square P_n$.
3. $\xi(P_n \square P_n) = n$.

Application of both zero forcing and minor monotonicity of ξ enables us to then calculate all rank spreads of $P_m \,\Box\, P_n,\ m \leq 3 \leq n$ over the real field, and many of the rank spreads of $P_m \,\Box\, P_n$ for $4 \leq m < n$. Whether or not $M(F, P_m \,\Box\, P_n) = \min\{m, n\}$ for every field F and the calculation of all the rank spreads of an $m \times n$ grid over the real field remain as yet open questions. We have also yet to find a subgraph G of $P_m \,\Box\, P_n$ for which the zero forcing number does not give the correct value of $M(F, G)$.

Minimum Rank of Looped Graphs with Cut Vertices

Rana Mikkelson (Iowa State University)

Minimum rank has been studied primarily for simple graphs, graphs that do not allow loops or multiple edges. For simple graphs a cut vertex reduction technique is known for calculating the minimum rank of graphs with cut vertices from some induced subgraphs. We extend this result to graphs that allow loops. Since our result holds over any field with more than two elements, we can use the result to show that the minimum rank of a looped tree has the same minimum rank over any field with three or more elements.

Two Topics in Combinatorial Matrix Theory & Graph Theory

Richard A. Brualdi

I plan to discuss two independent topics. One is concerned with graphs jointly arising from row and column orthogonality of a matrix. The other is concerned with a generalization of partitions of integers arising from graphs, with emphasis on trees.

Reduced Decompositions with Few Repetitions and Permutation Patterns

Dan Daly (Denver University)

A reduced decomposition of a permutation is a cycle decomposition using only cycles of the form $(i, i+1)$ which is minimal in length. Tenner, Stanley and others have started to connect the study of reduced decompositions with permutation patterns. We will discuss the pattern avoidance and containment conditions for permutations whose reduced decompositions have only one or two elements repeated and connections with the Bruhat order on the symmetric group.

A generalization of the concept of a k-plexe of latin squares
Kyle Pula (Denver University)

A k-plex of a latin square is a collection of cells representing each row, column, and symbol precisely k times. The classic example is that of a transversal, i.e. a 1-plex. I will introduce a weakening of the concept of a k-plex and give a construction showing that several existence conjectures for k-plexes hold at least in this weaker context. I will also show that the main non-existence result for k-plexes still applies for this weaker structure.

Combinatorial Matrix Theory and Spectral Graph Theory
Leslie Hogben (Iowa State University)

Structure Analysis of Acyclic Matrices by Special Vertices of Trees
In-Jae Kim (Minnesota State-Mankato)

To each n by n symmetric matrix $A = [a_{ij}]$ there is an associated graph $G(A)$ such that the vertices are $1, 2, \ldots, n$, and there is an edge between vertex i and vertex j $(i \neq j)$ if and only if $a_{ij} \neq 0$. If $G(A)$ is a tree, then A is acyclic. Let λ be an eigenvalue of A with multiplicity $m_A(\lambda)$, and $A(i)$ be the principal submatrix of A obtained by deleting row and column i from A. If $m_{A(i)}(\lambda) \geq m_A(\lambda)$ (resp. $m_{A(i)}(\lambda) > m_A(\lambda)$), then index i is said to be a Fiedler-vertex (resp. Parter-vertex) of A for λ. In this talk we provide geometric characterizations of Fiedler- and Parter-vertices of acyclic matrices. Furthermore, we describe a structure of an acyclic matrix by Fiedler- and Parter-vertices, which enables us to construct an acyclic matrix of a desired form according to the locations of Fiedler- and Parter-vertices. (This is a joint work with Bryan Shader)

Max Algebra Eigenvalues
Pauline van den Driessche (U. Victoria)

Introductions are given to max-plus algebra and max algebra, with emphasis on eigenequations and the relation between these in the two algebras. An algorithm is presented to compute the max eigenvalue and associated max eigenvector for an irreducible nonnegative square matrix A under a condition on its associated directed graph G(A). Using max algebra, bounds that depend on G(A) are derived for the Perron root of A.

Types Potential Convergence
of Modulus Patterns
O. Pryporova (Iowa State University)

In this talk, several types of potential convergence of modulus patterns that correspond to types of convergence of matrices will be introduced. A modulus pattern Z is convergent if all complex matrices in $Q(Z)$ (i.e. all matrices with modulus pattern Z) are convergent. Convergent modulus patterns have been completely characterized by E. Kaszkurewicz and A. Bhaya, however, a complete characterization of potential convergence is still an open question. It will be shown that potential absolute convergence and potential diagonal convergence are equivalent, and their complete characterization for $n \times n$ modulus patterns will be given. Complete characterizations of all introduced types of potential convergence for 2×2 modulus patterns and some examples for 3×3 modulus patterns will also be presented.

There is No Drake / Larson Linear Space on 30 Points
Anton Betten (Colorado State University)

A linear space is Drake / Larson if it contains at least two lines and there are no lines of size $2, 3$ or 6. The existence or nonexistence of such linear spaces on v points has been decided by Drake and Larson in 1983 for all $v \neq 30$. The same pair of authors address the open case on 30 points in "A quest for certain linear spaces on thirty points." *J. Statist. Plann. Inference*, 1984. The purpose of this talk is to settle this problem in the negative. The result relies on a combination of parameter calculation and exhaustive computer search. This work is in collaboration with Dieter Betten.

Embedding Partial Linear Spaces in Planes
Eric Moorhouse (University of Wyoming)

Every finite partial linear space embeds in a countably infinite projective plane, by a simple process of free closure. It is however open whether every finite partial linear space embeds in some finite projective (or affine) plane. Some appreciation for the difficulty of this problem is evident if we ask: which finite partial linear spaces are embeddable in a classical plane? (By 'classical' I mean Pappian, i.e. coordinatized by a field. The distinction between affine and projective planes matters very little in this context.) We may ask for an algorithm which, given a finite partial linear space, decides whether it is embeddable in a classical plane over some field; and if so, produces such an embedding. We show that this computational problem is at least as difficult as integer factorization. (Joint work with Jason Williford and John Hitchcock.)

Three-Bit Monomial Hyperovals
Timothy Vis (U. Colorado-Denver)

A set of $n+2$ points with no three on a line in a projective plane of order n is called a hyperoval. In a Desarguesian projective plane, every hyperoval can be represented by a polynomial. Currently, every hyperoval that can be represented by a monomial can be represented by a monomial whose exponent is the sum of at most three distinct powers of two. If the exponent is the sum of at most two distinct powers of two, the only hyperovals that can arise are the known ones. We discuss work to classify those hyperovals that are represented by a monomial whose exponent is the sum of exactly three distinct powers of two, as well as some ideas for a classification of all monomial hyperovals.

Extending Partial Tournaments

LeRoy Beasley*(Utah State University), David Brown (Utah State University) and Brooks Reid (Cal-State San Marcos)

Let A be a $(0, 1, *)$-matrix with main diagonal all 0's and such that if $a_{ij} = 1$ or then $a_{j,i} = *$ or 0. Under what conditions on the row sums, and or column sums, of A is it possible to change the $*$'s to 0's or 1's and obtain a tournament matrix (the adjacency matrix of a tournament digraph) with a specified score sequence? We answer this question in the case of regular and nearly regular tournaments. The result we give is best possible in the sense that no relaxation of any condition will always yield a matrix that can be so extended.

Spectrally arbitrary patterns over finite fields

Liz Bodine (Washington State University)

A zero-nonzero pattern $\mathcal{A}$ is spectrally arbitrary over a finite field $\mathbb{F}_q$ provided that for each monic polynomial $r(x) \in \mathbb{F}_q[x]$, there exists a matrix A over $\mathbb{F}_q$ with zero-nonzero pattern $\mathcal{A}$ such that the characteristic polynomial $p_A(x) = r(x)$. In this talk, we will investigate several zero-nonzero patterns over finite fields and show over which fields these patterns are spectrally arbitrary. We will explore the relationship between the number of nonzero entries and the size and characteristic of the finite field in order to develop some necessary conditions and sufficient conditions for a pattern to be spectrally arbitrary.
