§2.1 key. HW #2.

#3 $y'+y=te^{-t}+1$

(a) see the other page

(b) $y(t)\to 1$ as $t\to\infty$, from the graph.

(c) $\mu(t)=e^{t}$ (integrating factor)

$$\frac{d}{dt}(e^{t}y)=t+e^{t}$$

$$e^{t}y=\frac{1}{2}t^{2}+e^{t}+c$$

$$y=\frac{1}{2}t^{2}e^{-t}+1+ce^{-t}$$

when $t\to\infty$, $e^{-t}\to 0$. and the rate that $e^{-t}\to 0$ is faster than that of $t^{2}\to\infty$.
Hence $y(t)\to 1$. as $t\to\infty$

#5 $y'-2y=3e^{t}$

(a) see the other page

(b) $y(t)\to\pm\infty$, when $t\to\infty$. as shown in the graph.

(c) $\mu(t) = e^{-2t}$

$$\frac{d}{dt}\left(e^{-2t} y\right) = 3e^{-t}$$

$$e^{-2t} y = -3e^{-t} + c$$

$$y = -3e^{t} + c e^{2t}$$

It is clear that $y(t) \to \pm\infty$, (the sign is depending on c.)

#15 $\quad ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$

$$y' + \frac{2}{t} y = t - 1 + \frac{1}{t}$$

$$\mu(t) = \exp\left(2 \int \frac{1}{t} dt\right) = e^{2 \ln t} = e^{\ln t^2} = t^2$$

$$\frac{d}{dt}\left(t^2 y\right) = t^3 - t^2 + t$$

$$t^2 y = \frac{1}{4} t^4 - \frac{1}{3} t^3 + \frac{1}{2} t^2 + c$$

$$y(t) = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{c}{t^2}$$

$$y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c = \frac{1}{2} \Rightarrow c = \frac{1}{12}$$

$\therefore \quad y(t) = \frac{1}{4} t^2 - \frac{1}{3} t + \frac{1}{2} + \frac{1}{12 t^2}$ (fine here!)

$$= \left(3t^4 - 4t^3 + 6t^2 + 1\right) / \left(12 t^2\right)$$

#2

\# 21 $\quad y' - \frac{1}{2}y = 2\cos t \qquad y(0) = a$

(a) see the other page for the direction field.

The solution goes to either $+\infty$ or $-\infty$, when $t \to \infty$

(It is not clear from plots by "ODE Architect," but one can see the trent.)

(b) $\mu(t) = e^{-\frac{t}{2}}$

$$\frac{d}{dt}\left(e^{-t/2} y\right) = 2e^{-t/2}\cos t$$

$$I = \int 2e^{-t/2}\cos t\, dt \qquad \left(\begin{array}{ll} u = e^{-t/2}, & dv = 2\cos t\, dt \\ du = -\frac{1}{2}e^{-t/2}dt, & v = 2\sin t \end{array} \right)$$

$$= 2e^{-t/2}\sin t - \int -e^{-t/2}\sin t\, dt \qquad \text{(integration by parts)}$$

$$= 2e^{-t/2}\sin t + \int e^{-t/2}\sin t\, dt$$

$$= 2e^{-t/2}\sin t + \left[e^{-t/2}(-\cos t) - \int -\frac{1}{2}e^{-t/2}(-\cos t)\, dt \right]$$

$$= 2e^{-t/2}\sin t - e^{-t/2}\cos t - \frac{1}{4}I$$

$$5I = e^{-t/2}(8\sin t - 4\cos t)$$

$$I = e^{-t/2}\left(\frac{8}{5}\sin t - \frac{4}{5}\cos t\right)$$

Hence $\quad e^{-t/2} y = e^{-t/2}(8/5 \sin t - 4/5 \cos t) + c$

$$y = 8/5 \sin t - 4/5 \cos t + c e^{t/2}$$

$$y(0) = -4/5 + c = a \Rightarrow c = a + 4/5$$

$$\Rightarrow y(t) = 8/5 \sin t - 4/5 \cos t + (a + 4/5) e^{t/2}$$

(c)

It is clear that $y(t) \to$ either $+\infty$ or $-\infty$, when $t \to \infty$, depending on the sign of $(a + 4/5)$, if $(a + 4/5) \neq 0$.

However, when $a + 4/5 = 0$, or $a = a_0 = -4/5$

$$y(t) = 8/5 \sin t - 4/5 \cos t$$

and y becomes highly oscillatory when $t \to \infty$

Hence, when $a = a_0 = -4/5$, the behavior of the solution changes

(#3)

#25 $ty' + 2y = \sin t / t \qquad y(-\pi/2) = a, \quad t < 0$

(a) see the other page for the direction field.

From the graph, one can guess that when $t \to 0$, $y \to \pm\infty$

(b) $$y' + \frac{2}{t} y = \sin t / t^2$$

$$\mu(t) = \exp\left[\int \frac{2}{t} dt\right] = \exp[2 \ln t] = \exp[\ln(t^2)] = t^2$$

$$\frac{d}{dt}(t^2 y) = \sin t$$

$$t^2 y = -\cos t + c$$

$$y(t) = -\cos t / t^2 + c / t^2$$

$$y(-\pi/2) = c \cdot \frac{4}{\pi^2} = a, \quad \Rightarrow c = \frac{a\pi^2}{4}$$

Hence $y(t) = \left(-\cos t + \frac{a\pi^2}{4}\right) / t^2$

(c) If $(-\cos t + a\pi^2/4) \not\to 0$ when $t \to 0$, then it is clear that $y(t) \to \pm\infty$ when $t \to 0$, depending on the sign of $(-\cos t + a\pi^2/4)$

If $(-\cos t + a\pi^2/4) \to 0$ as $t \to 0$, then we have $\lim_{t \to 0} y(t) = \frac{0}{0}$, an indetermined form of $\frac{0}{0}$, We will need to find the limit with the L'Hôpital's rule.

Therefore when $a\pi^2/4 = 1$, or $a = a_0 = \frac{4}{\pi^2}$, we have

$$\lim_{t \to 0} y(t) = \lim_{t \to 0} \frac{-\cos t + 1}{t^2}$$

$$= \lim_{t \to 0} \frac{-\sin t}{2t}$$

$$= \lim_{t \to 0} \frac{\cos t}{2}$$

$$= \frac{1}{2}$$

#30

$$y' - y = 1 + 3\sin t \qquad y(0) = y_0$$

$$\mu(t) = e^{-t}$$

$$\frac{d}{dt}(e^{-t}y) = e^{-t} + 3e^{-t}\sin t$$

$$e^{-t}y = -e^{-t} - \frac{3}{2}e^{-t}\cos t - \frac{3}{2}e^{-t}\sin t + C$$ (see #21 for reference)

$$y(t) = -1 - \frac{3}{2}\cos t - \frac{3}{2}\sin t + ce^{t}$$

$$y(0) = -1 - \frac{3}{2} + c = y_0$$

$$c = y_0 + \frac{5}{2}$$

$$y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + \left(y_0 + \frac{5}{2}\right)e^{t}$$

If $y_0 + \frac{5}{2} \neq 0$, then $y(t) \to \infty$ as $t \to \infty$

Hence, for the solution remining finite as $t \to \infty$

we require $y_0 + \frac{5}{2} = 0$, or $y_0 = -\frac{5}{2}$

§2.2

#1

$$y' = x^2/y$$

$$yy' = x^2$$

$$\frac{d}{dx}\left(\frac{1}{2}y^2\right) = x^2$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

$$y^2 = \frac{2}{3}x^3 + C$$

or $3y^2 - 2x^3 = C$, $\quad y \neq 0$ (since $f(x,y) = x^2/y$)

#3.

$$y' + y^2 \sin x = 0$$

$$\sin x + \frac{1}{y^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d}{dx}\left[-\cos x + \frac{-1}{y}\right] = 0$$

or $-\cos x + \frac{-1}{y} = C$

$\Rightarrow y^{-1} + \cos x = C.$ if $y \neq 0$

Note: $y = 0$ is also a solution to the DE

#5

#5

$$y' = (\cos^2 x)(\cos^2 2y)$$

$\boxed{\text{case I}}$

$$\frac{1}{\cos^2 2y} y' = \cos^2 x \quad (\text{Note: } \cos 2y \neq 0)$$

$$\int \frac{1}{\cos^2 2y} \frac{dy}{dx} \cdot dx = \int \cos^2 x \, dx$$

LHS: $\displaystyle \int \frac{dy}{\cos^2 2y} \underset{\uparrow}{=} \frac{1}{2} \int \sec^2 z \, dz = \frac{1}{2} \tan z = \frac{1}{2} \tan(2y)$

$z = 2y$

$dy = \frac{1}{2} dz$

RHS: $\displaystyle \cos^2 x = \frac{1 + \cos 2x}{2}$

$$\int \cos^2 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)$$

Hence

$$\frac{1}{2} \tan(2y) = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

or

$$2\tan(2y) - 2x - \sin 2x = C, \quad \text{if } \cos 2y \neq 0.$$

$\boxed{\text{case II}}$

If $\cos 2y = 0 \Rightarrow y' = 0. \Rightarrow y$ is a constant so that $\cos 2y = 0, \Rightarrow y = \pm \frac{(2n+1)\pi}{4}.$

#7 $$\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^{y}}$$

If $y+e^{y} \neq 0$, then

$$(y+e^{y})\frac{dy}{dx} = x-e^{-x}$$

$$\int (y+e^{y})\frac{dy}{dx}\cdot dx = \int (x-e^{-x})\,dx$$

LHS: $$\int (y+e^{y})\,dy = \frac{1}{2}y^2+e^{y}$$

RHS: $$\int (x-e^{-x})\,dx = \frac{1}{2}x^2+e^{-x}$$

Hence

$$\frac{1}{2}y^2+e^{y}-\frac{1}{2}x^2 \overset{-}{\cancel{+}} e^{-x} = C$$

or

$$y^2+2e^{y}-x^2 \overset{-}{\cancel{+}} 2e^{-x} = C$$

or $(y^2-x^2)+2(e^{y}-e^{-x}) = C$, where $y+e^{y} \neq 0$

#9 $y' = (1-2x)y^2$; $y(0) = -1/6$

(#6)

(a) If $y \neq 0$

$$\frac{1}{y^2} y' = 1-2x$$

$$\int \frac{1}{y^2} y' \, dx = \int 1-2x$$

$$-y^{-1} = x - x^2 + c$$

$$y(0) = -\frac{1}{6} \Rightarrow c = 6$$

$$\frac{1}{y(x)} = x^2 - x - 6$$

or $y(x) = \dfrac{1}{x^2-x-6} = \dfrac{1}{(x-3)(x+2)}$

when $x \neq 3$, and $x \neq -2$

(What happen if $y=0$?) ← solution does not exist, since $0 \neq -\frac{1}{6}$.

(b) see the other page for the graph of the solution.

(c) y exist for $-2 < x < 3$. (the interval that contains $x = 0$)

#10 $y' = (1-2x)/y$ $\qquad y(1) = -2$

(a)

If $y \neq 0$, then

$$yy' = 1-2x$$

$$\Rightarrow \quad \frac{1}{2}y^2 = x - x^2 + c$$

$$y^2(x) = 2x - 2x^2 + c$$

$$y(x) = \pm\sqrt{2x-2x^2+c}$$

We cannot choose "+" sign, because the initial condition $y(1) = -2$, "$+\sqrt{}$" will not give us the desired initial data.

$$y(x) = -\sqrt{2x-2x^2+c}.$$

$$y(1) = -\sqrt{2-2+c} = -2, \quad c = 4.$$

Hence $y(x) = -\sqrt{2x-2x^2+4}$.

(b) See the other page for the graph of the solution.

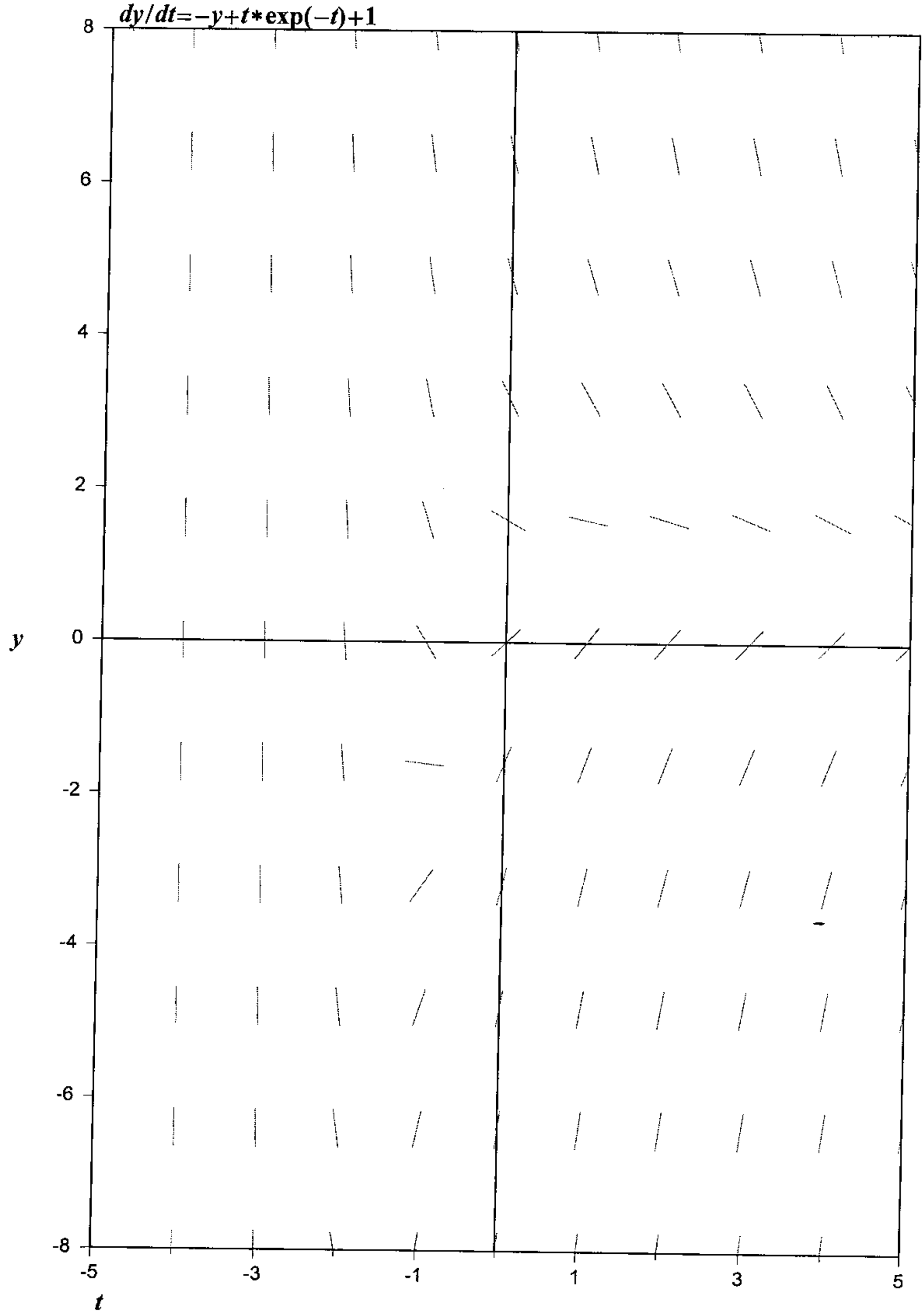
(cannot be "$\geq$", since $y \neq 0$)

(c) Since $2x-2x^2+4 > 0 \Rightarrow x^2-x-2 < 0 \Rightarrow -1 < x < 2$.

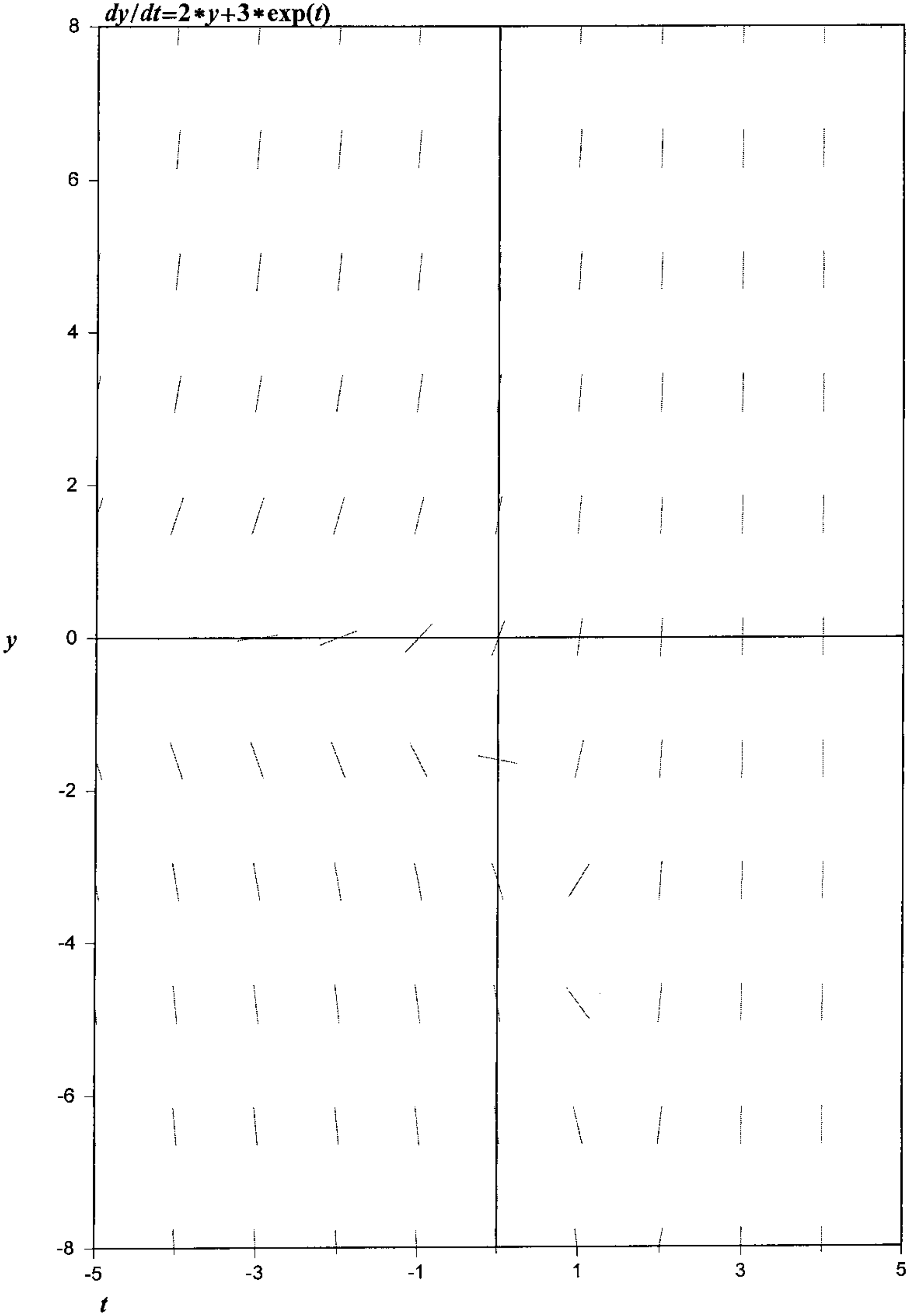
The initial point $x=1$ is contained in $(-1, 2)$

§2.1 # 3.

#7

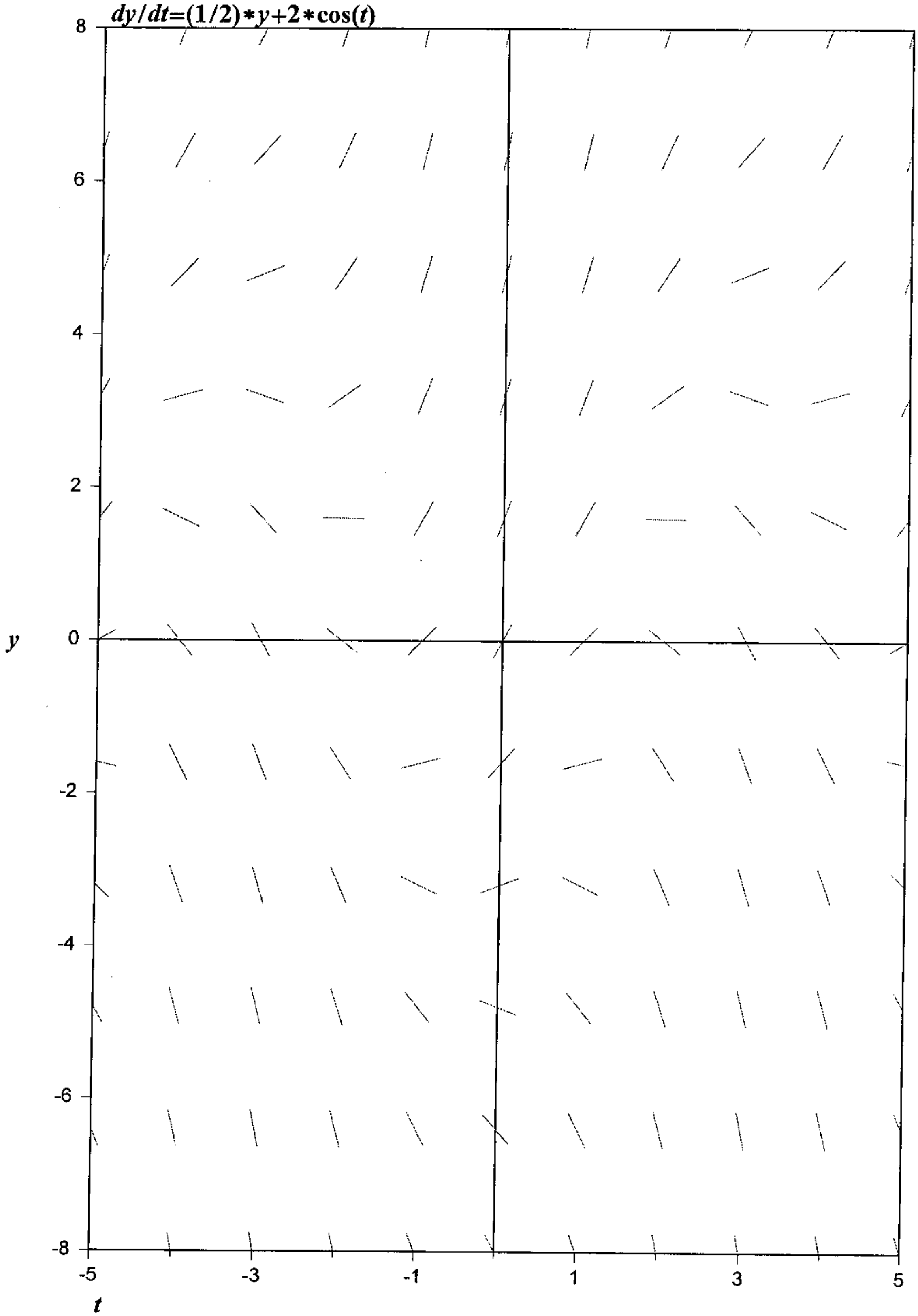


§2.1 #5

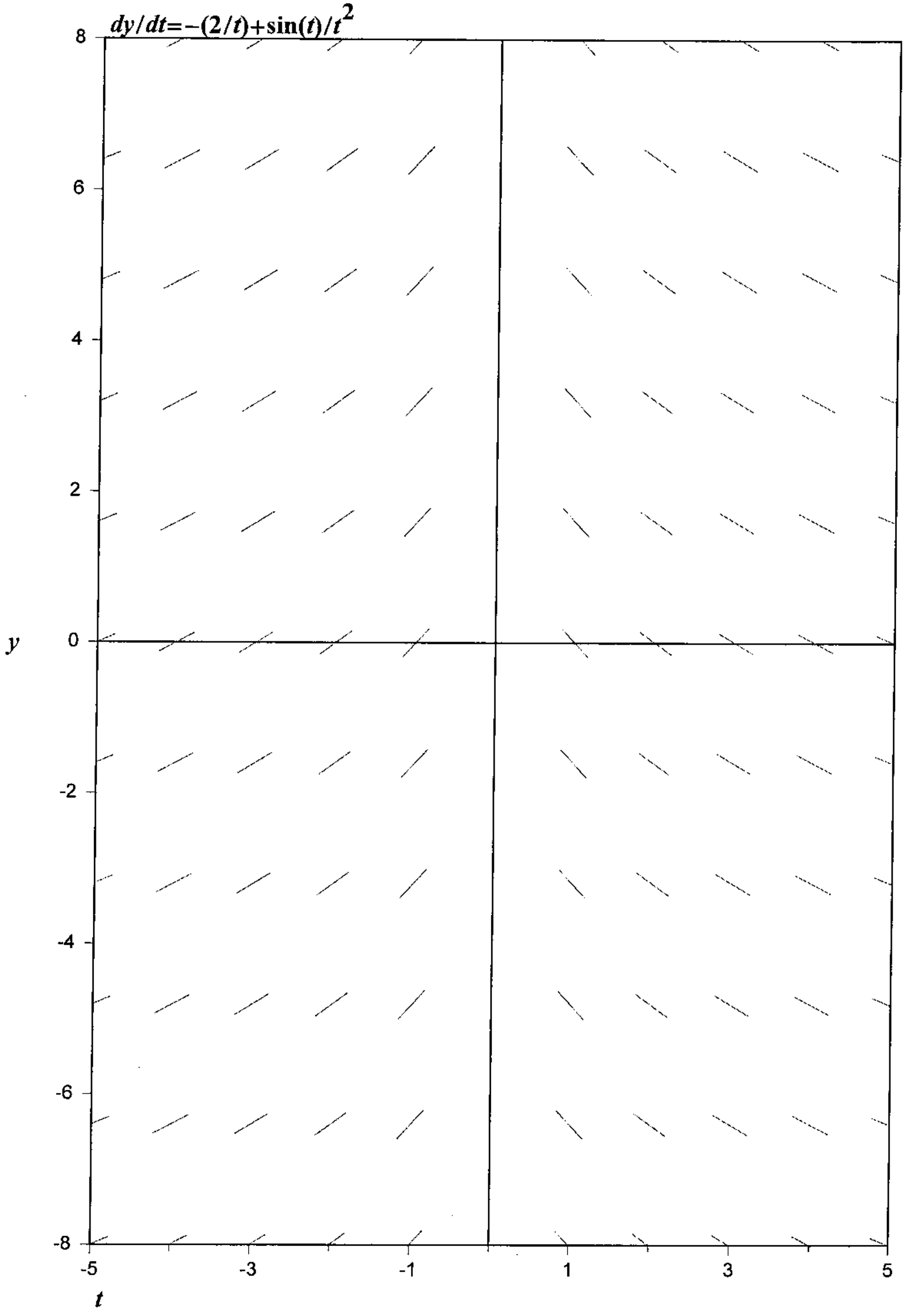


§2.1 #21

#8



§2.1 #25



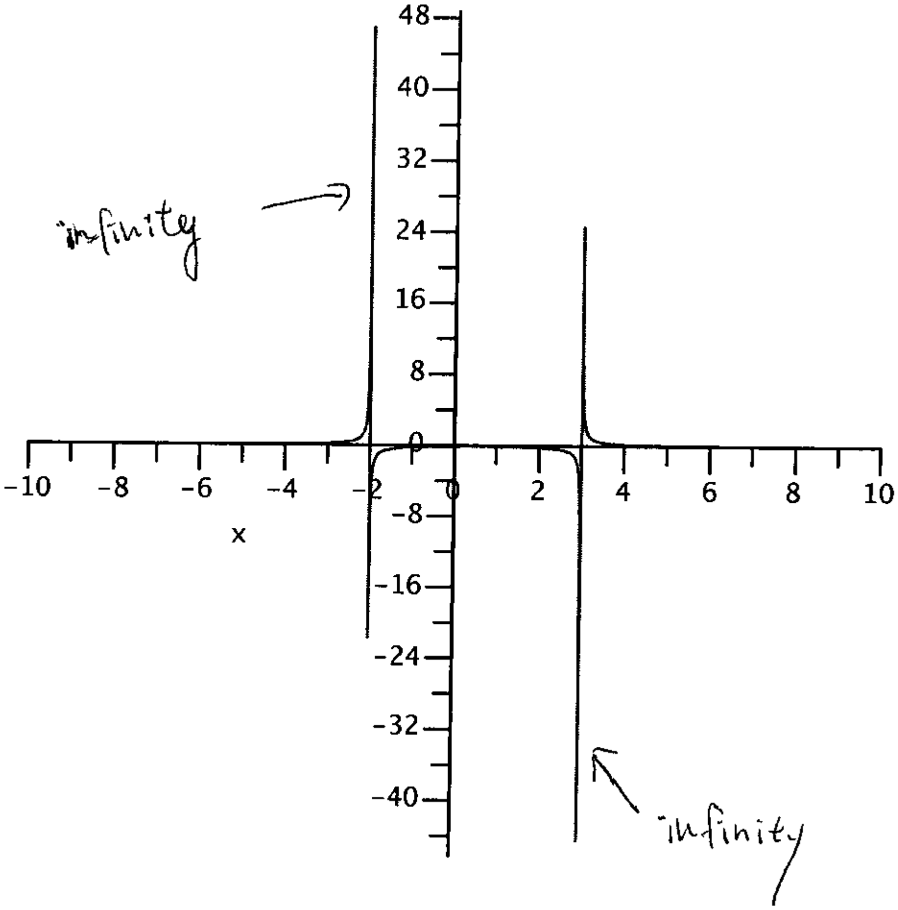
§2.2 #9

#9

$f := \frac{1}{((x-3)\cdot(x+2))};$

$$\frac{1}{(x-3)\,(x+2)} \tag{1}$$

$plot(f, x = -10..10);$



§ 2.2 #10

$f := -sqrt(2 \cdot x - 2 \cdot x^2 + 4);$

$$-\sqrt{2\,x - 2\,x^2 + 4} \qquad (1)$$

$plot(f, x = -0.9999 .. 1.999);$