MATH 5530 – SOLUTIONS TO HW#3

#1. Let $g \in G$, and write $y = x^g \in X$. Since H is transitive, there exists $h \in H$ such that $x^h = y$, so $x^{gh^{-1}} = y^{h^{-1}} = x$, i.e. $gh^{-1} \in G_x$. Now $g \in G_x h$, which shows that $G \subseteq G_x H \subseteq G$ and $G = G_x H$.

#4. Let $H \le S_4$ be a transitive subgroup. Then $4 \,\big|\, |H| \,\big|\, 24$ so $|H| \in \{4, 8, 12, 24\}$. The possibilities for H, up to conjugacy in S_4 (and therefore to within equivalence as permutation groups) are:

(i) $H = \langle (12)(34), (13)(24) \rangle$, elementary abelian of order 4, acting regularly on 4 points;

(ii) $H = \langle (1234) \rangle$, cyclic of order 4, also regular on 4 points;

(iii) $H = \langle (1234), (13) \rangle$, dihedral of order 8;

(iv) $H = A_4$ of order 12; or

(v) $H = S_4$ of order 24.

Note that S_4 has unique subgroups of order 12 and 24, giving cases (iv) and (v) respectively. Also S_4 has exactly three Sylow 2-subgroups, all of which are conjugate; this gives case (iii).

In every other case, $|H| = 4$ so H is regular; and there are just two groups of order 4, listed as (i) and (ii) above.

#2. Let $X = X_1 \cup X_2 \cup \cdots \cup X_t$ be the partition of X into G-orbits. For each $x \in X_i$ we have $[G : G_x] = |X_i|$. Count in two different ways the cardinality of the set $S = \{(x, g) \in X \times G : x^g = x\}$, as follows:

$$|S| = \sum_{x \in X} |G_x| = \sum_{i=1}^{t} \sum_{x \in X_i} |G_x| = \sum_{i=1}^{t} \sum_{x \in X_i} \frac{|G|}{|X_i|}$$

$$= \sum_{i=1}^{t} |X_i| \cdot \frac{|G|}{|X_i|} = \sum_{i=1}^{t} |G| = t|G|.$$

On the other hand,

$$|S| = \sum_{g \in G} |\{x \in X : x^g = x\}| = \sum_{g \in G} |\mathrm{Fix}(g)|.$$

Equating these two expressions for $|S|$ gives

$$t = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|,$$

the average number of fixed points for elements of G, as required.

#3. In #2 we have $t = 1$, so $\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| = 1$.

On average, elements of G fix 1 point. Since the identity fixes more than one point (recall that $|G| > 1$ so $|X| > 1$), there exists at least one $g \in G$ fixing fewer points than the average, i.e. $|\mathrm{Fix}(g)| < 1$, so $|\mathrm{Fix}(g)| = 0$.

#15. Let $X = \mathbb{Z}^2 = \{(x,y) : x, y \in \mathbb{Z}\}$.

The group $\mathbb{Z} \wr \mathbb{Z}$ is generated by the permutations of X which we denote

$$\sigma(x,y) = \begin{cases} (x+1, 0), & \text{if } y = 0; \\ (x,y), & \text{if } y \neq 0; \end{cases}$$

$$\tau(x,y) = (x, y+1).$$

Now $\mathbb{Z} \wr \mathbb{Z} = \langle \sigma, \tau \rangle$ is finitely generated! (just 2 generators). Let H be the subgroup

$$H = \{ g \in \langle \sigma, \tau \rangle : g(x,y) \text{ has second coordinate } y \}$$

This is the kernel of the action on the heights of points; hence $H \triangleleft \mathbb{Z} \wr \mathbb{Z}$.

Every element of H shifts only a finite number of rows, i.e. every $h \in H$ has the form

$$h(x,y) = (c_y + x, y) \quad \text{for all } (x,y) \in X$$

where $c_y \in \mathbb{Z}$ is zero for all but finitely many $y \in \mathbb{Z}$. In particular for all $h \in H$, there exists an integer $m_h \geq 0$ such that $h(x,y) = (x,y)$ whenever $|y| \geq m_h$. If $h_1, \dots, h_n \in H$ and $h \in \langle h_1, \dots, h_n \rangle$, then $h(x,y) = (x,y)$ whenever

$|y| \geq \max\{m_{h_1}, m_{h_2}, \cdots, m_{h_n}\}$. But there exists $h \in H$ not satisfying this condition: take

$$h(x,y) = \begin{cases} (x,y) & \text{if } y \neq y_0 \,; \\ (x+1, y_0) & \text{if } y = y_0 \end{cases}$$

where $y_0 \in \mathbb{Z}$ is chosen such that $y_0 > \max\{m_{h_i}\}$. This shows that no finite subset $\{h_1, \cdots, h_n\} \subset H$ can generate H.