Two-Graphs
and
Skew Two-Graphs

Two-graphs $\leftrightarrow$ Switching-equivalence classes of ordinary graphs

Skew Two-graphs $\leftrightarrow$ Switching-equivalence classes of tournaments

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Problem: Given translation planes $\pi_1, \pi_2, \ldots, \pi_n$ of order $n$, determine the isomorphism classes.

$n=16$ Dempwolf, Reifart (1983)

$n=25$ Oakden, Czerwinski (1992)

$n=49$ Mathon, Royle; Chames, Dempwolf (1992)

$n=27$ Dempwolf (1994)

J.H. Conway's invariant ($n$ odd):

plane $\pi$ \hspace{1cm} \rightarrow \hspace{1cm} \text{Fingerprint } f(\pi)$

(a sequence of integers) computable in $O(n^3)$ operations

$$\pi \rightarrow \begin{cases} \text{two-graph } \Delta(\pi), & n \not\equiv 3 \mod 4 \\ \text{(trivial if } g \text{ even)} \\ \text{skew two-graph } \nabla(\pi), & n \equiv 3 \mod 4 \end{cases}$$

\text{degree sequence } \rightarrow \text{ fingerprint } f(\pi)$

In all known cases, $f(\pi') = f(\pi) \iff \pi', \pi$ isomorphic or polar for $n \equiv 1 \mod 4$

This fails badly for $n \equiv 3 \mod 4$.

Eg $n=27$, $\exists \pi_1, \pi_2, \ldots, \pi_7$, $f(\pi_7) = f(\pi_5) = f(\pi_3)$. Dempwolf introduced Ken vector (computable in $O(n^4)$ time) to distinguish them.
$|X| = v \geq 3$

A two-graph is a $\Delta \leq \binom{X}{3}$ such that every 4-subset of $X$ contains an even number (i.e. 0, 2 or 4) of triples from $\Delta$.

E.g. the trivial two-graphs $\{\emptyset\}$ (empty) $\{\binom{X}{3}\}$ (complete)

$\Delta$ two-graph $\Rightarrow \overline{\Delta} := \binom{X}{3} \setminus \Delta$ complementary two-graph

The degree of $\{x, y, z\} \subset X$ is the number of triples $\{x, y, z\} \in \Delta$ containing $\{x, y, z\}$.

$\Delta$ is regular $\iff$ every 2-subset $\{x, y\} \subset X$ has the same degree $\iff \Delta$ is a 2-$(v, 3, \lambda)$ design, some $\lambda$

$\Gamma$ ordinary graph with vertex set $X$.

For $X_1 \subseteq X$, $\Gamma(X_1)$ is the graph formed by replacing edges between $X_1$ and $X \setminus X_1$ by nonedges.

$\Gamma, \Gamma'$ are switching-equivalent $\iff \Gamma' = \Gamma(X_1)$, some $X_1 \subseteq X$.

$\iff A' = DAD$, some $\pm 1$-diagonal matrix $D$ where $A, A'$ are the $(0, \pm 1)$-adjacency matrices of $\Gamma, \Gamma'$ (-1 for adj., +1 for nonadj., 0 on diag.)

$\Delta(\Gamma) := \{ \{x, y, z\} \in \binom{X}{3} : \Gamma$ contains an odd number (i.e. 1 or 3) of $\{x, y, z\}, \{x, z\}, \{y, z\}$

$\Delta(\Gamma) = \Delta(\Gamma') \iff \Gamma, \Gamma'$ are switching equivalent

$\Delta(\overline{\Gamma}) = \overline{\Delta(\Gamma)}$
$|X| = v \geq 3$

$\text{Sym}(X) = \{ \text{permutations of } X \}$

$\mathcal{J}(X) = \{ 3\text{-cycles } (x, y, z) \in \text{Sym}(X) \}$

$|\mathcal{J}(X)| = \frac{(v-1)(v-2)}{2}$

A skew two-graph (Cameron, 1977): oriented two-graph

is a subset $\mathcal{D} \subseteq \mathcal{J}(X)$ such that

(i) $\forall \tau \in \mathcal{J}(X)$, exactly one of $\tau, \tau'$ is in $\mathcal{D}$;

(ii) $\forall \{x, y, z, w\} \in \binom{X}{4}$, $\mathcal{D}$ contains an even number (i.e. 0, 2 or 4) of the 3-cycles $(x, y, z), (x, w, y), (x, z, w), (y, w, z)$.

(The latter is a conjugacy class of $\text{Alt}(\{x, y, z, w\})$.)

$\overline{\mathcal{D}} := \{ \tau': \tau \in \mathcal{D} \}$ is the complementary skew two-graph.

$\overline{\mathcal{D}}$ is a trivial skew two-graph.

The degree of an ordered pair $(x, y)$ in $X$, is the number of $z \in X$ such that $(x, y, z) \in \mathcal{D}$.

$\mathcal{D}$ is regular $\iff$ every pair $(x, y)$ has the same degree, necessarily $\frac{v-2}{2}$.

A tournament $T$ on $X$ is an orientation of the complete graph on $X$.

For $X \leq X$, a tournament $T(X_i)$ is obtained by reversing all edges between $X_i$ and $X - X_i$.

$T, T'$ switching-equivalent $\iff$ $T' = T(X_i), \text{ some } X_i \leq X$

$\iff A' = DAD$, some $\pm 1$-diagonal matrix $D$

where $A, A'$ are the $(0, \pm 1)$-adjacency matrices of $T, T'$ (skew-symmetric).

$\mathcal{D}(T) := \{ (x, y, z) \in \mathcal{J}(X) : T \text{ contains an odd no. (i.e. 1 or 3) of } (x, y), (y, z), (z, x) \}$

$\mathcal{D}(T) = \mathcal{D}(T') \iff T, T'$ switching equivalent

$\mathcal{D}(T) = \overline{\mathcal{D}(T)}$

$\mathcal{D}(T)$ regular $\iff$ A skew-symmetric conference matrix

$\iff A + I \text{ (skew) Hadamard matrix}$

$\Rightarrow v \equiv 0 \mod 4$
\( \mathcal{P} \): classical finite polar space embedded in \( PG(V) \).
\( f(\cdot,\cdot) \): associated bilinear/sesquilinear form.
Elements of \( \mathcal{P} \) are totally isotropic (or tot. singular)
subspaces of \( V \):
- points, lines, ..., m-flats, ..., generators.

A cap \( \mathcal{O} \) in \( \mathcal{P} \) is a set of points, no two collinear
in \( \mathcal{P} \) (perp. with respect to \( f \)).
\( \mathcal{O} \) is an ovoid if each generator meets \( \mathcal{O} \) in a
unique point.

**Theorem:** If \( \mathcal{P} \) is of \( \begin{cases} \text{orthogonal} \\ \text{unitary} \\ \text{symplectic, } q \equiv 3 \mod 4 \end{cases} \) type, then
\[ \Delta(\mathcal{O}) := \{ \{ \omega_1, \omega_2, \omega_3 \in \mathcal{O} \} : f(\omega_1, \omega_2) f(\omega_2, \omega_3) f(\omega_3, \omega_1) = [\mathcal{O}] \} \]
is a two-graph (trivial if \( q \) even).

If \( \mathcal{P} \) is of symplectic type, \( q \equiv 3 \mod 4 \), then
\[ \Delta(\mathcal{O}) := \{ \{ \omega_1, \omega_2, \omega_3 \in \mathcal{O} \} : f(\omega_1, \omega_2) f(\omega_2, \omega_3) f(\omega_3, \omega_1) = [\mathcal{O}] \} \]
is a skew two-graph.

Isometries preserve \( \Delta(\mathcal{O}) \) (or \( \Delta(\mathcal{O}) \)).
Similarities preserve \( \Delta(\mathcal{O}) \) (or \( \Delta(\mathcal{O}) \)) to within complementation.

Eq. Paley two-graphs \( \Delta_q \quad (q \equiv 1 \mod 4) \)
and skew two-graphs \( \Delta_q \quad (q \equiv 3 \mod 4) \)
from the \( Sp(2,q) \) ovoids:

\( F = GF(q), \quad q \) odd
\( \mathcal{O} = X = PG(1,F) \) projective line
\( G = Sp(V,f) = Sp(2,F) \cong SL(2,q) \)
acts 2-transitively on \( X \); two orbits on-triples
\( \langle \omega_1, \omega_2, \omega_3 \rangle \), distinguished according to whether
or not \( f(\omega_1, \omega_2) f(\omega_2, \omega_3) f(\omega_3, \omega_1) \) is a square.

If \( q \equiv 1 \mod 4 \) then \( \Delta_q := \Delta(X,f) \) is one \( G \)-orbit
on \( (X)_3 \); the other is \( \overline{\Delta_q} \cong \Delta_q \)
regular two-graph of degree \( \frac{q-1}{2} \).

If \( q \equiv 3 \mod 4 \) then \( \Delta_q := \Delta(X,f) \) is one \( G \)-orbit
on \( (X)_3 \); the other is \( \overline{\Delta_q} \cong \Delta_q \).
Taylor (1992) classified the 2-transitive two-graphs.

**Theorem.** Every 2-transitive skew two-graph \( V \) is isomorphic to \( \Delta_2 \) (Paley type), some \( q \equiv 3 \mod 4 \).

**Proof.** Let \( G = \text{Aut} \, V \), and let \( g \in G \) involution.

If \( g \) fixes \( x \in X \) then \((x, y, z) \overset{g}{\rightarrow} (x, z, y)\) for some \( y, z \in X \). But only one of \( \tau, \tau^{-1} \) is in \( V \), contradiction. So involutions in \( G \) are fixed-point-free.

Bender (1968) \( \Rightarrow \) \( G \cong \text{PSL}(3, q) \), \( X = \text{PG}(1, q) \), \( q \equiv 3 \mod 4 \).

\( \Rightarrow \) Both orbits of \( G \) on \( X \) are isomorphic to \( \Delta_2 \).

Kleidman (1988) classified the 2-transitive ovoids.

If \( O \) is a 2-transitive ovoid then \( \Delta(O) \) is 2-transitive or trivial (or \( V(O) \) is 2-trans.)

<table>
<thead>
<tr>
<th>Ovoid</th>
<th>Restrictions</th>
<th>Nontrivial ( \Delta ) or ( V )?</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Sp(2,q) )</td>
<td>( q \equiv 1 \mod 4 )</td>
<td>nontrivial ( \Delta )</td>
<td>Paley ( \Delta_q )</td>
</tr>
<tr>
<td>( Sp(2,q) )</td>
<td>( q \equiv 3 \mod 4 )</td>
<td>nontrivial ( V )</td>
<td>Paley ( V_q )</td>
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<tr>
<td>( O_7(q) )</td>
<td>( q ) odd</td>
<td>trivial ( \Delta )</td>
<td>Theorem 7.3</td>
</tr>
<tr>
<td>( U(3,q) )</td>
<td>( q ) odd</td>
<td>nontrivial ( \Delta )</td>
<td>unitary two-graph</td>
</tr>
<tr>
<td>( O_2^+(q) )</td>
<td>( q ) odd</td>
<td>trivial ( \Delta )</td>
<td>( 1 + \frac{q}{2} ) siml. classes</td>
</tr>
<tr>
<td>( O_2^+(q) )</td>
<td>( q ) odd</td>
<td>nontrivial ( \Delta )</td>
<td>Paley ( \Delta_q^2 )</td>
</tr>
<tr>
<td>( O_6(q), O_4^2(q) )</td>
<td>( q ) odd</td>
<td>nontrivial ( \Delta )</td>
<td>induced from ( O_4(q) )</td>
</tr>
<tr>
<td>( U(4,q) )</td>
<td>( q ) odd</td>
<td>nontrivial ( \Delta )</td>
<td>induced from ( U(3,q) )</td>
</tr>
<tr>
<td>( O_7(3) )</td>
<td>( q = 3 )</td>
<td>nontrivial ( \Delta )</td>
<td>( Sp(6,2) )</td>
</tr>
<tr>
<td>( O_8(q) )</td>
<td>( q = 3^s ), ( s ) odd</td>
<td>nontrivial ( \Delta )</td>
<td>( PSU(3, q) )</td>
</tr>
<tr>
<td>( O_7(q) )</td>
<td>( q = 3^s ), ( s ) odd</td>
<td>nontrivial ( \Delta )</td>
<td>( 2G_2(q) )</td>
</tr>
<tr>
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<td>( q = 3^s )</td>
<td>nontrivial ( \Delta )</td>
<td>induced from ( O_7(q) )</td>
</tr>
</tbody>
</table>

Our construction of the unitary two-graphs from the \( U(3,q) \) ovoid follows [Sel1]. The non-triviality of \( \Delta(O) \) for the last four entries, follows from Theorem 7.4; hence by [Ta2], these are the usual unitary and Ree two-graphs. All remaining cases are covered by remarks above and Theorems 7.2 and 7.3.

Suppose \( I(O) \) is an invariant of caps \( O \) which is computed by testing just \( k \)-subsets of \( O \), and that the invariant \( I \) is nontrivial (able to distinguish at least two inequivalent caps of the same size). In the octagonal case, Theorem 4.3 shows that \( k \geq 3 \), and if \( k = 3 \), then \( I(O) \) is a function of \( \Delta(O) \) and the characteristic is odd. [Note: It is usually possible to define a nontrivial invariant graph \( I(O) \); for example, fix \( t \geq 0 \) and let \( I(O) \) be the set of pairs \( (u, \langle \rangle) \) in \( O \) such that \( |r \cap O| = t \) for some plane \( r \) of \( PG(V) \) containing \( (u, v) \). However, the latter definition evidently requires testing subsets of \( O \) of size \( \geq 4 \).]

For symplectic polar spaces, however, there are nontrivial triple-based invariants in even characteristic, computed by testing for collinear triples (cf. Theorem 4.4).

4.3 Theorem. Let \( P \) be an orthogonal polar space in \( PG(V) = PG(s, F) \), \( s \geq 2 \), with associated quadratic form \( Q \) on \( V \). Then the number of orbits of \( PG(V, Q) \) on ordered 3-caps in \( P \) is

(i) \( 1 \), if \( q \) is even and \( s \) is odd;

(ii) \( 2 \), if \( q \) is odd and \( P \neq \theta(4,q) \) (the orbit containing \( (u, \langle \rangle, \langle \rangle) \) being determined by whether \( f(u,v)f(v,w)f(w,u) \) is a square or a nonsquare in \( F \)); or
Theorem. Suppose $\mathcal{P}$ is of orthogonal type, but not $O_q^+(2)$. The number of orbits of $PSL(V,Q)$ on caps of size 3 is

1. if $q$ even;
2. if $q$ odd, distinguished by whether or not $f(u,v)f(v,w)f(w,u)$ is a square.

\[
\Delta = \Delta(\Gamma) \quad \text{or} \quad \nabla = \nabla(\Gamma)
\]

\[
A = (\alpha, \pm 1) - \text{adjacency matrix of } \Gamma \text{ or } T \quad \rightarrow \quad f(\Delta) \quad \text{or} \quad f(\nabla)
\]

\[
f(\Delta) \triangledown := \text{multiset of entries of } |AA^T|,
\]

\[
f(\nabla)
\]

Theorem. If $n_\lambda = \text{no. of } \{(x,y)\} \text{ of degree } \lambda \text{ in } \nabla$

\[
f(\Delta) = \begin{cases} 
2^{n_{r+1}} & \text{if } v = 2r+1 \\
2^{n_{r+1}} & \text{if } v = 2r-1 \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \end{cases}
\]

\[
f(\nabla) = \begin{cases} 
0 & \text{if } v = 2r+1 \\
2^{n_{r+1}} & \text{if } v = 2r-1 \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \\
2^{n_{r+1}} & \text{if } v = 2r \end{cases}
\]
A partial m-system in $\mathcal{P}$ is a collection $M = \{ \pi_1, \pi_2, \ldots, \pi_k \}$ of m-flats s.t. $\pi_i^\perp \cap \pi_j = \emptyset$ for all $i \neq j$.

**Theorem (Shult, Thas)** There exists an upper bound for $k = |M|$ depending on $\mathcal{P}$ but not on $m$.

If $|M|$ attains this upper bound, $M$ is an $m$-system.

<table>
<thead>
<tr>
<th>partial 0-system</th>
<th>=</th>
<th>cap</th>
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</thead>
<tbody>
<tr>
<td>0-system</td>
<td>=</td>
<td>ovoid</td>
</tr>
<tr>
<td>partial r-system</td>
<td>=</td>
<td>partial spread</td>
</tr>
<tr>
<td>r-system</td>
<td>=</td>
<td>spread</td>
</tr>
</tbody>
</table>

$\{ \nu_{i_0}, \nu_{i_1}, \ldots, \nu_{i_m} \}$ basis for $\pi_i$

$\text{sgn} (\pi_i, \pi_j) := \text{sgn} \det \left[ f(\nu_{i_0}, \nu_{j_0}), f(\nu_{i_0}, \nu_{j_1}), \ldots, f(\nu_{i_0}, \nu_{j_m}) \right]_{0 \leq \alpha, \beta \leq m} = \pm 1$

**Theorem** If $\mathcal{P}$ is orthagonal, unitary, symplectic, $q^{m+1} \not\equiv 3 \mod 4$ then

$\Delta(M) := \{ \{ \pi_i, \pi_j, \pi_k \} : \text{sgn}(\pi_i, \pi_j) \text{sgn}(\pi_j, \pi_k) \text{sgn}(\pi_k, \pi_i) = -1 \}$

is a two-graph (trivial if $q$ even).

If $\mathcal{P}$ is of symplectic type, $q^{m+1} \equiv 3 \mod 4$,

similarly get a skew two-graph $\nabla(M)$.

$\Delta(M)$ (or $\nabla(M)$) invariant under isometries.

$\Delta(M)$ (or $\nabla(M)$) invariant (to within complementation) under similarities.
J.H. Conway's Description

\( \pi \): translation plane of order \( n \)

\[ 1 \quad 2 \quad 3 \ldots \quad i \ldots \quad n+1 \]

O

\[ a_{ij} = \text{sign of this permutation} \]

\( a_{ii} = 0 \)

A = \((a_{ij})_{1 \leq i, j \leq n+1}\)

Fingerprint \( f(\pi) \) = multiset of entries of \( |AA^T| \).

Theorem

Let \( \{M_1, M_2, \ldots, M_n\} \) be a spread set for \( \pi \). Then WLOG

\[ a_{ij} = \begin{cases} 
\text{sgn} \det(M_i - M_j), & 1 \leq i, j \leq n \\
1, & i < n+1 = j \\
\pm 1, & j < n+1 = j \\
0, & i = j = n+1 
\end{cases} \]

* choose 5 if \( n \equiv 3 \mod 4 \)

\[ l-1 \text{ if } n \equiv 3 \mod 4 \]

Theorem

If \( \Omega \) is an ovoid in \( O^+_6(q) \) and \( \pi \) is the corresponding translation plane of order \( q^2 \), then \( \Delta(\Omega) = \Delta(\pi) \) to within complementation. So \( f(\Omega) = f(\pi) \).
How large can a cap $\mathcal{O}$ in $\mathcal{P}$ be if $\Delta(\mathcal{O})$ is trivial? What structure of $\mathcal{O}$ attains this maximum?

**Theorem.** A cap of type $\mathcal{O}_5(q)$, $q$ odd, $S = \text{disc}(\mathcal{O})$.

$\mathcal{O}$ cap in $\mathcal{P}$.

(i) $\Delta(\mathcal{O})$ complete and $-2S = \emptyset \Rightarrow |\mathcal{O}| = q+1$.

(ii) $\Delta(\mathcal{O})$ empty and $-2S = \emptyset \Rightarrow |\mathcal{O}| \leq q+1$.

Moreover, $\mathcal{O}$ is a BLT-set $\Leftrightarrow$ equality holds in (i) or (ii).

**Recall:** A BLT-set in $\mathcal{O}_5(q)$ is a collection of $q+1$ singular points s.t. $\langle u, v, w \rangle^\perp$ is an elliptic (anisotropic) line $V$ distinct $\langle u \rangle, \langle v \rangle, \langle w \rangle$ in $\mathcal{O}$.

BLT-set $\mathcal{O}$ $\Leftrightarrow$ q-clan $\Leftrightarrow$ Flock of quadratic cone in $PG(3, q)$

**Theorem.** Let $\mathcal{O}$ be a $(q+1)$-cap in $O_q^-(q^*)$, $q$ odd.

Then $\Delta(\mathcal{O})$ trivial $\iff$ $\mathcal{O}$ conic.

**Theorem.** $(q+1)$-cap in $O_q^+(q^*), q = p^e$ odd.

Then $\Delta(\mathcal{O})$ trivial $\iff$ $\mathcal{O}$ 2-transitive

(e iso. classes, $1 + \frac{e}{2}$ simil. classes)

(BLT $\mathcal{O} \Leftrightarrow$ linear or Kantor flock)


These results are equivalent to Thas (1987) in flock language.
Theorem: A polar space is naturally embedded in $\text{PG}(n, q)$, where $q = p^e$ and $p$ is odd.

For a cap in $\mathcal{Q}$, $\Delta(\mathcal{Q})$ trivial $\Rightarrow |\mathcal{Q}| \leq \binom{n + \frac{p-1}{2}}{n}^e + 1$.

$\varphi \in V^*$

$v \in V$

Classical Circle Geometries

$\varphi(v) q^{-1}$ has $p$-rank $\left(\binom{n + p - 1}{n}\right)^e$

Mobius (inversive) plane

$\varphi \in V^*$

$v \in V$

has $p$-rank $\left(\binom{n + \frac{p-1}{2}}{n}\right)^e$

Laguerre plane

Minkowski plane
\( E = \{ C_1, C_2, \ldots, C_k \} \) **doubly intersecting circles** in Laguerre plane, i.e. \( |C_i \cap C_j| = 2 \ \forall i \neq j \)

**Theorem** There is a family of \( (3q-1)/2 \) doubly intersecting circles.

**Theorem** \( k = |E| \leq \frac{q^2 + 1}{2} \)

(cf. Blokhuis and Bruen (1989) for Miquelian inversive plane)

**Theorem** \( |E| \leq \left( \frac{q^{q+1}}{4} \right)^e \) where \( q = p^2 \) odd.

\[ |E| \leq \begin{cases} q^{1.47} & p = 3 \\ q^{1.68} & p = 5 \\ q^{1.83} & p = 7 \end{cases} \]