Homework # 2
Math 5200-Spring 2009
Due date February 19th 2009.

1. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. A (set) function $\mu : \mathcal{A} \longmapsto [0, \infty]$ is called a measure on $\mathcal{A}$ if it satisfies the following properties:
   (i) $\mu(\emptyset) = 0$.
   (ii) Whenever $\{A_n\}_n$ is a disjoint sequence of $\mathcal{A}$
   $$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$
   holds, that is μ is σ-additive.
   - Let X be a set and let $\mathcal{A} = \mathcal{P}(X)$. Define $\mu_1 : \mathcal{A} \longmapsto [0, \infty]$ by $\mu_1(A) = \infty$ if A is an infinite subset of X and $\mu_1(A)$=the number of elements of A if A is a finite set. Prove that μ_1 is a measure on $\mathcal{A}$.
   - Fix an element $a \in X$ and define $\mu_2 : \mathcal{A} \longmapsto [0, \infty]$ by $\mu_2(A) = 0$ if $a \notin A$ and $\mu_2(A) = 1$ if $a \in A$. Prove that μ_2 is a measure on $\mathcal{A}$.
2. Prove that any function defined on a set of measure zero is measurable.
3. Let $f(x)$ be a measurable and $g(x)$ be monotone increasing (i .e. $g(x)$ is such that $g(x_1) \geq g(x_2)$ for $x_1 > x_2$). Prove that $g(f(x))$ is measurable.
4. Investigate the measurability of the function
   $$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number in } (0,1) \\ 0 & \text{if } x \text{ is an irrational number in } (0,1) \end{cases}$$