Homework # 6
Math 5200-Spring 2009
Due date April 9th 2009.

1. **(ex3, pp194)** Note that

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + 2Re < f, g >$$

for any pair of elements in a Hilbert space H. As a result verify the identity

$$\|f+g\|^2 + \|f-g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right).$$

2. **(ex5, pp194)** Establish the following relations between $L^2(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$:

   (a) Neither the inclusion $L^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ nor the inclusion $L^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ is valid.

   (b) Note, however, that if f is supported on a set E of finite measure and if $f \in L^2(\mathbb{R}^d)$, applying the Cauchy-Schwarz inequality to $f\chi_E$ gives $f \in L^1(\mathbb{R}^d)$, and

$$\|f\|_{L^1(\mathbb{R}^2)} \leq m(E)^{1/2} \|f\|_{L^2(\mathbb{R}^2)}.$$

   (c) If f is bounded ($|f(x)| \leq M$), and $f \in L^1(\mathbb{R}^d)$ then $f \in L^2(\mathbb{R}^d)$, with

$$\|f\|_{L^2(\mathbb{R}^2)} \leq M^{1/2} \|f\|^{1/2}_{L^1(\mathbb{R}^2)}.$$

   Hint: For (a) consider $f(x) = |x|^{-\alpha}$, when $|x| \leq 1$ or when $|x| > 1$.

3. **(ex7, pp194)** Suppose $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. Prove that the collection $\{\varphi_{k,j}\}_{k,j=1}^{\infty}$ with $\varphi_{k,j}(x,y) = \varphi_k(x)\varphi_j(y)$ is an orthonormal basis of $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.
Hint: First verify that $\{\varphi_{k,j}\}$ are orthonormal, by Fubini's theorem. Next for each j consider

$$F_j(x) = \int_{\mathbb{R}^d} F(x,y)\overline{\varphi_j(y)}dy.$$

If one assumes that

$$< F, \varphi_{k,j} >= 0, \text{ forall } j$$

then

$$\int F_j(x)\overline{\varphi_k(x)}dx = 0.$$

4. If $f \in L^1$ and $g \in L^\infty$ prove that

$$\int |fg| \leq \|f\|_1 \|g\|_\infty,$$

where $\|\cdot\|_p$ is the norm in L^p.