Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
I. (15 points) In a certain college, 25% of the students failed mathematics (M), 15% of the students failed chemistry (C) and 10% of the students failed both mathematics and chemistry. A student is selected at random.

1. If he failed chemistry, what is the probability that he failed mathematics?
2. If he failed mathematics, what is the probability that he failed chemistry?
3. What is the probability that he failed mathematics or chemistry?

**Solution:** Let us denote by

\[ M = \{ \text{Students who failed Mathematics} \}, \]
\[ C = \{ \text{Students who failed Chemistry} \}, \]

then \( P(M) = 0.25, \ P(C) = 0.15, \) and \( P(M \cap C) = 0.10. \)

1. The probability that a student failed Mathematics given that he (she) failed Chemistry is

\[ P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{2}{3}. \]

2. The probability that a student failed Chemistry given that he (she) failed Mathematics is

\[ P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{0.10}{0.25} = \frac{2}{5}. \]

3. The probability that a student failed Chemistry or Mathematics is

\[ P(C \cup M) = P(M) + P(C) - P(M \cap C) = 0.25 + 0.15 - 0.10 = 0.3 = \frac{3}{10}. \]
II. (15 points) Let \( X \) be a discrete random variable such that
\[
P(X = a) = p \quad \text{and} \quad P(X = b) = 1 - p.
\]

1. Show that \( \frac{X - b}{a - b} \) is a Bernoulli random variable.

2. Find \( \text{Var}(X) \).

Solution:

1. Let us compute the following probability mass function for the r.v. \( Y \).

\[
P(Y = 0) = P \left( \frac{X - b}{a - b} = 0 \right)
= P(X - b = 0)
= P(X = b)
= 1 - p.
\]

And

\[
P(Y = 1) = P \left( \frac{X - b}{a - b} = 1 \right)
= P(X - b = a - b)
= P(X = a)
= p.
\]

Hence, we get that \( Y \) takes values 0 and 1 with respectively probability \( 1 - p \) and \( p \). Hence, \( Y \) is a Bernoulli.

2. Since \( Y \) is a Bernoulli random variable, then
\[
\text{Var}(Y) = p(1 - p).
\]

Moreover,

\[
\text{Var}(Y) = \text{Var} \left( \frac{X - b}{a - b} \right)
= \frac{1}{(a - b)^2} \text{Var}(X - a)
= \frac{1}{(a - b)^2} \text{Var}(X)
\]

This implies that
\[
\text{Var}(X) = (a - b)^2 \text{Var}(Y) = (a - b)^2 p(1 - p)
\]
III. (20 points) Let $X$ and $Y$ be independent random variables taking values in the positive integers and having the same mass function $f(n) = \frac{1}{2^n}$ for $n = 1, 2, \ldots$. Find

1. $P(\min\{X,Y\} \leq k)$.
2. $P(Y > X)$.
3. $P(X > kY)$ for a given positive integer $k$.

Hint: Recall some results from geometric series

1. $\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$ for $|r| < 1$.

Solution:

1. 
\[
P(\min (X,Y) \leq k) = 1 - P(\min (X,Y) > k) = 1 - P(X > k,Y > k)
= 1 - P(X > k) P(Y > k) = 1 - [P(X > k)]^2 = 1 - \left[ \sum_{m=k+1}^{\infty} P(X = m) \right]^2
\]

\[
1 - \left[ \sum_{m=k+1}^{\infty} \frac{1}{2^m} \right]^2 = 1 - \left[ \frac{1}{2^{k+1}} \sum_{m=0}^{\infty} \frac{1}{2^m} \right]^2
\]

On the other side the infinite geometric series $\sum_{m=0}^{\infty} \frac{1}{2^m} = 2$. Hence

\[
P(\min (X,Y) \leq k) = 1 - \left[ \frac{1}{2^k} \right]^2.
\]
2. Using the partition equation by conditioning on \( X \), we have that

\[
P(Y > X) = \sum_{m=1}^{\infty} P(Y > X | X = m) P(X = m)
\]

\[
= \sum_{m=1}^{\infty} P(Y > m | X = m) P(X = m)
\]

\[
= \sum_{m=1}^{\infty} P(Y > m) P(X = m)
\]

\[
= \sum_{m=1}^{\infty} \sum_{l=m+1}^{\infty} P(Y = l) P(X = m)
\]

\[
= \sum_{m=1}^{\infty} \sum_{l=m+1}^{\infty} \frac{1}{2^m} \frac{1}{2^l}
\]

\[
= \sum_{m=1}^{\infty} \frac{1}{2^m} \left( \sum_{l=0}^{\infty} \frac{1}{2^l} - \sum_{l=0}^{m} \frac{1}{2^l} \right)
\]

On the other side \( \sum_{l=0}^{\infty} \frac{1}{2^l} = 2 \) and \( \sum_{l=0}^{m} \frac{1}{2^l} = \frac{1-(1/2)^{m+1}}{1-1/2} \). Hence \( \sum_{l=0}^{m} \frac{1}{2^l} = 2 - \frac{1}{2^m} \) and

\[
P(Y > X) = \sum_{m=1}^{\infty} \frac{1}{2^m} = \frac{1}{3}.
\]

3. Using again the partition equation by conditioning on \( Y \), we get that

\[
P(X > kY) = \sum_{y=1}^{\infty} P(X > kY | Y = y) P(Y = y)
\]

\[
= \sum_{y=1}^{\infty} P(X > ky | Y = y) P(Y = y)
\]

\[
= \sum_{y=1}^{\infty} P(X > ky) P(Y = y)
\]

\[
= \sum_{y=1}^{\infty} P(Y = y) \sum_{l=ky+1}^{\infty} P(X = l)
\]

On the other side using the index shift and the infinite geometric series we get that

\[
\sum_{l=ky+1}^{\infty} P(X = l) = \frac{1}{2ky}.
\]
Hence,

\[
P(X > kY) = \sum_{y=1}^{\infty} \frac{1}{2y} \frac{1}{2^{ky}}
\]

\[
= \sum_{y=1}^{\infty} \frac{1}{2^{(k+1)y}} = \frac{1}{2^{(k+1) - 1}}
\]
VI. \textbf{(20 points)} Let $X$ and $Y$ two random variables jointly distributed with a density given by

$$f_{(X,Y)}(x, y) = \begin{cases} 
  x(y - x)e^{-y} & 0 \leq x \leq y < \infty \\
  0 & \text{otherwise}
\end{cases}$$

1. Show that:
   
   (a) \quad f_{X|Y}(x|y) = 6x(y - x)y^{-3}, \quad 0 \leq x \leq y

   (b) \quad f_{Y|X}(y|x) = (y - x)e^{x-y}, \quad 0 \leq x \leq y < \infty.

2. Deduce that

   (a) \quad E(X|Y) = \frac{1}{2} Y

   (b) \quad E(Y|X) = X + 2

\textbf{Solutions:}

1. First, let us compute the marginals $f_X$ and $f_Y$.

   For $y \geq x$:

   $$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$
   $$= \int_{0}^{y} x(y - x)e^{-y}dx$$
   $$= \int_{x}^{\infty} (xy - x^2)e^{-y}dx$$
   $$= e^{-y}(yx^2/2 - x^3/3)|_{0}^{y}$$
   $$= e^{-y}y^3/6.$$

   Hence, for $y \geq x$

   $$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$
   $$= \frac{x(y - x)e^{-y}}{e^{-y}y^3/6}$$
   $$= 6x(y - x)y^{-3}.$$
(b) For $0 < x \leq y$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$= \int_{x}^{\infty} x(y - x)e^{-y} dy$$

$$= x \left[ (y - x)(-e^{-y})|_{x}^{\infty} + \int_{x}^{\infty} e^{-y} dy \right]$$

$$= x(-e^{-y})|_{x}^{\infty}$$

$$= xe^{-x}.$$ 

Hence, for $0 < x \leq y$

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(y)}$$

$$= \frac{x(y - x)e^{-y}}{xe^{-x}}$$

$$= (y - x)e^{x-y}.$$ 

2. (a)

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

$$= \int_{x}^{\infty} x[6x(y - x)y^{-3}] dx$$

$$= \frac{1}{2}y.$$ 

Hence, we deduce that $E(X|Y) = \frac{1}{2}Y$.

(b)

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy$$

$$= \int_{0}^{y} y(y - x)e^{x-y} dy$$

$$= x + 2.$$ 

Hence, we deduce that $E(Y|X) = X + 2$
V. (20 points) Suppose that $X$ and $Y$ are independent and identically distributed (iid) with a standard normal distributions $N(0,1)$.

1. Compute the density of $Z := \sqrt{2}Y$

2. Compute the joint density of $(X, Z)$.

3. Compute the moment generating function of $X + Z$

Solution:

1. $X$ and $Y$ have the same distribution that we denote by $F$ and density which is given by

   $$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$  

   Let us compute the distribution of $\sqrt{2}Y$:

   $$F_Z(y) := P(\sqrt{2}Y \leq y)$$
   $$= P(Y \leq \frac{y}{\sqrt{2}})$$
   $$= F\left(\frac{y}{\sqrt{2}}\right).$$

   Hence,

   $$f_Z(y) = \frac{d}{dy} (F_Z(y))$$
   $$= \frac{d}{dy} \left( F\left(\frac{y}{\sqrt{2}}\right) \right)$$
   $$= \frac{1}{\sqrt{2}} f\left(\frac{y}{\sqrt{2}}\right)$$
   $$= \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}}$$

   Hence, $Z \sim N(0,2)$

2. Since $X$ and $Y$ are independent then $X$ and $\sqrt{2}Y$ are also independent. Hence, the joint density of $X$ and $Z$ is given by the product of their densities that is:

   $$f_{X,Z}(x, y) = f_X(x)f_Z(y)$$
   $$= \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) \left( \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} \right)$$
   $$= \frac{1}{2\pi \sqrt{2}} e^{-\frac{x^2}{2} - \frac{y^2}{4}}.$$
3. $X \sim N(0, 1)$ and $Z \sim N(0, 2)$ and $X$ and $Z$ are independent, hence their sum $X + Z \sim N(0, 3)$. Hence, the MGF of their sum is given by

$$M_{X+Z}(t) = e^{\frac{3}{2}t^2}.$$
VI. (10 points) Let $X$ and $Y$ two random variables identically distributed and not necessary independent. Compute

$$Cov(X + Y, X - Y)$$

**Solution:**


$$= Cov(X, X) - Cov(X, Y) + Cov(Y, X) - Cov(Y, Y)$$

Since $Cov(X, Y) = Cov(Y, X)$

$$= Cov(X, X) - Cov(X, Y) + Cov(X, Y) - Cov(Y, Y)$$

$$= Cov(X, X) - Cov(Y, Y)$$

$$= Var(X) - Var(Y)$$

Now, $X$ and $Y$ are identically distributed, hence they have the same variance

$$Var(X) = Var(Y).$$

Hence,

$$Cov(X + Y, X - Y) = 0.$$