Partial credit will be awarded for your answers, so it is to your advantage to explain your reasoning and what theorems you are using when you write your solutions. Please answer the questions in the space provided and show your computations.

Good luck!
I. (10pts) In a certain college, 25% of the students failed mathematics, 15% of the students failed chemistry and 10% of the students failed both mathematics and chemistry. A student is selected at random.

1. If he failed chemistry, what is the probability that he failed mathematics?
2. If he failed mathematics, what is the probability that he failed chemistry?
3. What is the probability that he failed mathematics or chemistry?

Solution: Let us denote by

\[ M = \{ \text{Students who failed Mathematics} \} , \]
\[ C = \{ \text{Students who failed Chemistry} \} , \]

then \( P(M) = 0.25, \ P(C) = 0.15, \) and \( P(M \cap C) = 0.10. \)

1. The probability that a student failed Mathematics given that he (she) failed Chemistry is

\[ P(M|C) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{2}{3}. \]

2. The probability that a student failed Chemistry given that he (she) failed Mathematics is

\[ P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{0.10}{0.25} = \frac{2}{5}. \]

3. The probability that a student failed Chemistry or Mathematics is

\[ P(C \cup M) = P(M) + P(C) - P(M \cap C) = 0.25 + 0.15 - 0.10 = 0.3 = \frac{3}{10}. \]
II. (10pts) Let \( X \) and \( Y \) be independent random variables taking values in the positive integers and having the same mass function \( f(k) = \frac{1}{2^k} \) for \( k = 1, 2, \ldots \). Find

1. \( P(\min\{X, Y\} \leq k) \).
2. \( P(Y > X) \).
3. \( P(X > kY) \) for a given positive integer \( k \).

Solution:

1.

\[
P(\min (X, Y) \leq k) = 1 - P(\min (X, Y) > k) = 1 - P(X > k, Y > k)
= 1 - P(X > k) P(Y > k) = 1 - [P(X > k)]^2 = 1 - \left[ \sum_{m=k+1}^{\infty} P(X = m) \right]^2
= 1 - \left[ \sum_{m=k+1}^{\infty} \frac{1}{2m} \right]^2 = 1 - \left[ \frac{1}{2^{k+1}} \sum_{m=0}^{\infty} \frac{1}{2m} \right]^2
\]

On the other side the infinite geometric series \( \sum_{m=0}^{\infty} \frac{1}{2m} = 2 \). Hence

\[
P(\min (X, Y) \leq k) = 1 - \left[ \frac{1}{2^k} \right]^2.
\]

2.

\[
P(Y > X) = \sum_{m=1}^{\infty} P(Y > m, X = m)
= \sum_{m=1}^{\infty} P(Y > m) P(X = m)
= \sum_{m=1}^{\infty} \sum_{l=m+1}^{\infty} P(Y = l) P(X = m)
= \sum_{m=1}^{\infty} \sum_{l=m+1}^{\infty} \frac{1}{2m} \frac{1}{2^l}
= \sum_{m=1}^{\infty} \frac{1}{2^m} \left( \sum_{l=0}^{\infty} \frac{1}{2^l} - \sum_{l=0}^{m+1} \frac{1}{2^l} \right)
\]

On the other side \( \sum_{l=0}^{\infty} \frac{1}{2^l} = 2 \) and \( \sum_{l=0}^{m+1} \frac{1}{2^l} = \frac{1-(1/2)^{m+1}}{1-1/2} \). Hence \( \sum_{l=0}^{m+1} \frac{1}{2^l} = \frac{1}{2^m} \) and

\[
P(Y > X) = \sum_{m=1}^{\infty} \frac{1}{2^{2m}} = \frac{1}{3}.
\]
3.

\[ P(X > kY) = \sum_{y=1}^{\infty} P(X > ky, Y = y) \]

\[ \sum_{y=1}^{\infty} P(X > ky) P(Y = y) \]

\[ \sum_{y=1}^{\infty} P(Y = y) \sum_{l=ky+1}^{\infty} P(X = l) \]

On the other side using the index shift and the infinite geometric series we get that

\[ \sum_{l=ky+1}^{\infty} P(X = l) = \frac{1}{2^{ky}}. \]

Hence,

\[ P(X > kY) = \sum_{y=1}^{\infty} \frac{1}{2y} \frac{1}{2^{ky}} \]

\[ = \sum_{y=1}^{\infty} \frac{1}{2^{(k+1)y}} = \frac{1}{2^{(k+1)} - 1} \]
III. (10pts)
Let $X$ be a discrete random variable such that

$$X(\omega) = \begin{cases} 
1 & \text{ if } \omega \in A \\
2 & \text{ otherwise}
\end{cases}$$

1. Find the distribution function of $X$ if $P(A) = \frac{1}{3}$.
2. Find the expected and the variance of $X$.

Solution:

1. The distribution function of $X$ is defined as

$$F_X(x) = P(X \leq x).$$

If $x < 1$ then $P(X \leq x) = 0$.
If $1 \leq x < 2$ then $P(X \leq x) = P(X = 1) = P(A) = \frac{1}{3}$.
If $x \geq 2$ then $P(X \leq x) = P(X = 1 \text{ or } X = 2) = P(A) + P(A^c) = 1$.

Hence

$$F_X(x) = \begin{cases} 
0 & x < 1 \\
\frac{1}{3} & 1 \leq x < 2 \\
1 & x \geq 2
\end{cases}$$

2. 

$$E(X) = \sum_k kP(X = k) = 1P(X = 1) + 2P(X = 2)$$

$$= P(A) + 2P(A^c) = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}.$$ 

On the other side

$$E(X^2) = \sum_k k^2P(X = k) = 1^2P(X = 1) + 2^2P(X = 2)$$

$$= P(A) + 4P(A^c) = \frac{1}{3} + \frac{4}{3} = 3.$$ 

Hence,

$$Var(X) = E(X^2) - (E(X))^2 = 3 - \left(\frac{5}{3}\right)^2 = \frac{2}{9}.$$
VI. (10pts) Suppose that $X$ and $Y$ are jointly continuous random variables with joint density

$$f_{(X,Y)}(x,y) = \begin{cases} 
    ce^{x+y} & (x, y) \in (-\infty, 0] \times (-\infty, 0] \\
    0 & \text{otherwise}
\end{cases}$$

1. what is the value of $c$?
2. What is the probability that $X < Y$?
3. What are the marginal densities $f_X$ and $f_Y$?

Solution:
1. $f_{(X,Y)}$ is a joint density if $f_{(X,Y)} \geq 0$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)dxdy = 1$.
   Hence
   - $c \geq 0$
   - 
     $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)dxdy = c \int_{-\infty}^{0} \int_{-\infty}^{0} e^{x+y}dxdy = c \left( \int_{-\infty}^{0} e^x dx \right)^2 = c.$$  
   Hence $c = 1$.
2. Let us denote by $D = \{(x, y) \in \mathbb{R}^2, x < y\}$
   $$P(X < Y) = P((X, Y) \in D) = \int \int_D f_{(X,Y)}(x,y)dxdy$$
   $$= \int_{-\infty}^{0} dy \int_{-\infty}^{y} e^{x+y}dxdy = \int_{-\infty}^{0} e^{2y}dy = \frac{1}{2}$$
3. $X$ and $Y$ are symmetric hence If $x > 0$ then $f_X = f_Y = 0$
   If $x \leq 0$ then
   $$f_Y = f_X = \int_{-\infty}^{\infty} f_{(X,Y)}(x,y)dy = \int_{-\infty}^{0} e^{x+y}dy = e^x.$$  
   Hence
   $$f_X(x) = \begin{cases} 
    e^x & x \leq 0 \\
    0 & x > 0,
\end{cases}$$
   $$f_Y(y) = \begin{cases} 
    e^y & y \leq 0 \\
    0 & y > 0.
\end{cases}$$
V. (10pts) If $X$ is a Poisson random variable with parameter $\lambda$, show that

1. $E[X^n] = \lambda E[(X + 1)^{n-1}]$.

2. Now use this result to compute $E[X^3]$.

Solution:

1. 

$$E(X^n) = \sum_{k=0}^{\infty} k^n P(X = k) = \sum_{k=0}^{\infty} k^n \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} k^n \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=1}^{\infty} k^{n-1} \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} k^{n-1} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda \sum_{m=0}^{\infty} (m + 1)^{n-1} \frac{\lambda^m}{m!} e^{-\lambda} = \lambda \sum_{m=0}^{\infty} (m + 1)^{n-1} P(X = m) = \lambda E((X + 1)^{n-1}).$$

2. Let us take $n = 3$ in the above equality, we get that

$$E(X^3) = \lambda E((X + 1)^2) = \lambda E[X^2 + 2X + 1] = \lambda [E(X^2) + 2E(X) + 1].$$

On the other side $E(X) = Var(X) = \lambda$ and $E(X^2) = Var(X) + (E(X))^2 = \lambda + \lambda^2$. Hence,

$$E(X^3) = \lambda^3 + 3\lambda^2 + \lambda.$$
VI. (5pts (Bonus)) $X$ is a Poisson random variable with parameter $\lambda$, Show for $r = 2, 3, \ldots$,

$$E[X(X-1) \ldots (X-r+1)] = \lambda^r.$$