Notes on the $2n$-conjecture*

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1 Introduction

A *sign pattern* is an $n \times n$ matrix, $\mathcal{A}$, with entries in $\{+, -, 0\}$. If A is a real $n \times n$ matrix for which each entry has the same sign as its corresponding entry in $\mathcal{A}$, then A is a *realization* of $\mathcal{A}$, and we write $A \in \mathcal{A}$. The $2n$-conjecture is related to the study of the spectral properties among the matrices in $\mathcal{A}$.

The $n \times n$ sign pattern $\mathcal{A}$ is a *spectrally arbitrary pattern* (or a SAP, for short), provided that for each real, monic polynomial $r(x)$ of degree n there is a realization $A \in \mathcal{A}$ whose characteristic polynomial $p_A(x)$ is $r(x)$. Equivalently, $\mathcal{A}$ is a SAP provided each conjugate-closed multi-set of n complex numbers is the spectrum of at least one realization of $\mathcal{A}$.

As an example, consider the sign-pattern

$$\mathcal{A} = \begin{bmatrix} + & - \\ + & - \end{bmatrix}$$

and an arbitrary real polynomial $r(x) = x^2 + ux + v$. It is easy to verify that

$$\begin{bmatrix} |u| + |v| + 1 & -1 \\ v + (|u| + |v| + 1)(|u| + |v| + 1 + u) & -(|u| + |v| + 1 + u) \end{bmatrix}$$

has sign pattern $\mathcal{A}$ and characteristic polynomial $r(x)$. Hence $\mathcal{A}$ is a SAP. More

*Please feel free to e-mail me with corrections and suggestions.

generally, in [BMOV04],

$$\mathcal{V}_n = \begin{bmatrix} + & - & & & & O \\ + & & - & & & \\ \vdots & & & \ddots & & \\ + & & O & & - & \\ + & & & & & - \\ + & & & & & - \end{bmatrix}$$

is shown to be a SAP for $n \geq 3$, by explicitly showing that each characteristic polynomial can be realized.

Each 2×2 sign pattern whose diagonal vector is one of $(+,+)$, $(+,0)$, $(0,+)$, or $(0,0)$ is not a SAP, as the coefficient of the linear term in the characteristic polynomial of each realization of such a sign pattern will be non-positive. Similarly each 2×2 sign pattern whose diagonal entries are $(-,-)$, $(-,0)$, $(0,-)$ or $(0,0)$ is not a SAP. Each 2×2 sign pattern with at least one off-diagonal entry equal to zero is not a SAP, as the constant terms of the characteristic polynomials of such realizations don't take on each possible sign.

Since their introduction in [DJOV00] , there has been a flurry of research on SAPs. Research has focused on construction of SAPs [BMOV04, CV05, MTV05, CKSV05], the classification of small order SAPs and the combinatorial structure of SAPs [BMOV04, CM06, DHHMMPSV06]. The $2n$-conjecture concerns the latter line of research, and stems from the question:

What is the smallest number of nonzero entries in an $n \times n$ SAP?

The following theorem shows that an irreducible[1] $n \times n$ sign pattern has at least $2n - 1$ nonzero entries.

Theorem 1 *If $\mathcal{A}$ is an irreducible $n \times n$ SAP with $n \geq 2$, then A has at least $2n - 1$ nonzero entries.*

In light of Theorem 1, the smallest number of nonzero entries in an **irreducible** SAP of order n $(n \geq 2)$ is either $2n - 1$ or $2n$. Several families of n by n SAPs with exactly $2n$ nonzero entries are known (see Section 2). At this time there are no known examples of $n \times n$ SAPs with $2n - 1$ nonzero entries. This leads to the $2n$-conjecture (first stated in [BMOV04]).

Conjecture 1 *For each integer $n \geq 2$, the smallest number of nonzero entries in an $n \times n$ SAP is $2n$.*

[1] Recall that an $n \times n$ matrix with $n \geq 2$ is *reducible* if it is permutationally similar to a matrix of the form

$$\begin{bmatrix} B & O \\ C & D \end{bmatrix},$$

where B and D are square (nonvacuous) matrices. A matrix is *irreducible* if it is not reducible.

In other words, the $2n$-conjecture proposes that no $n \times n$ sign pattern with less than $2n$ nonzero entries is a SAP. In [BMOV04, DHHMMPSV06], it is shown that the conjecture is true for $n \leq 5$. Kevin Vander Meulen's student, Kevin West, is currently working on the $n = 6$ case, and appears to have a lengthly proof that the conjecture is valid for $n = 6$.

The only known proofs of Theorem 1 are algebraic. We include a proof of this theorem to illustrate techniques whose refinement might lead to the resolution of the conjecture. This proof is similar to the one in [BMOV04], but better lends itself to generalizations.

The proof depends on a lemma about diagonal scalings of irreducible matrices. Let $A = [a_{ij}]$ be a real, $n \times n$ matrix. Given a set $S = \{(i_1, j_1), \ldots, (i_p, j_p)\}$ of positions of A whose entries are nonzero, we define G_S to be the graph with vertices $1, 2, \ldots, n$ and edges joining i_k and j_k for $k = 1, 2, \ldots, p$. We say that G_S is a *tree of* A provided G_S is a connected, acyclic graph that spans the vertex set $\{1, 2, \ldots, n\}$. The *digraph of* A consists of the vertices $1, 2, \ldots, n$, and the arcs (i, j) for which $a_{ij} \neq 0$. Note that if A is irreducible, and S consists of the arcs of a breadth-first search of the digraph of A, then G_S is a tree of A.

Lemma 2 *Let $A = [a_{ij}]$ be an irreducible, $n \times n$ matrix, and let $S = \{(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})\}$ be a set of positions of A whose entries are nonzero such that G_S is a tree of A. Then there exist positive real numbers $d_1, \ldots, d_n$ such that the (i_k, j_k)-entry of DAD^{-1} equals $a_{i_k,j_k}/|a_{i_k,j_k}|$, where $D = diag(d_1, \ldots, d_n)$.*

Proof. Without loss of generality we may assume that the vertices $1, 2, \ldots, n$ are ordered so that exactly one of i_k and j_k belong to $\{1, 2, \ldots, k\}$ for $k = 1, \ldots, n-1$. We define the d_i's recursively. Set $d_1 = 1$. Assume that $d_1, \ldots, d_{k-1}$ have been defined and $2 \leq k \leq n$. If $i_k \notin \{1, 2, \ldots, k\}$, then set $d_{i_k} = d_{j_k}/|a_{i_k,j_k}|$, and if $j_k \notin \{1, 2, \ldots, k\}$ set $d_{j_k} = d_{i_k}|a_{i_k,j_k}|$. Setting D to be $\text{diag}(d_1, d_2, \ldots, d_n)$ we see that the (i_k, j_k)-entry of DAD^{-1} is $d_{i_k}a_{i_k,j_k}/d_{j_k} = (d_{j_k}/|a_{i_k,j_k}|)(a_{i_k,j_k}/d_{j_k}) = a_{i_k,j_k}/|a_{i_k,j_k}|$ if $i_k \notin \{1, 2, \ldots, k\}$, and is $d_{i_k}a_{i_k,j_k}/d_{j_k} = d_{i_k}a_{i_k,j_k}/(d_{i_k}|a_{i_k,j_k}|) = a_{i_k,j_k}/|a_{i_k,j_k}|$ if $j_k \notin \{1, 2, \ldots, k\}$. The result follows. ■

Throughout the remainder of these notes, let $X_1, X_2, \ldots,$ be distinct indeterminates, and let $\mathbb{R}[X_1, \ldots, X_n]$ denote the ring of polynomials in $X_1, \ldots, X_n$. Also, for $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}[X_1, \ldots, X_n]$, let $\mathbb{R}[\alpha_1, \ldots, \alpha_k]$ denote the subring of $\mathbb{R}[X_1, \ldots, X_n]$ generated by $\alpha_1, \ldots, \alpha_k$. The list $\alpha_1, \ldots, \alpha_k$ is *algebraically independent* provided $p(\alpha_1, \ldots, \alpha_k) \neq 0$ for every nonzero polynomial $p \in \mathbb{R}[X_1, \ldots, X_k]$. A well-known result (see [I94]) asserts that no list of $n + 1$ or more elements of $\mathbb{R}[X_1, X_2, \ldots, X_n]$ is algebraically independent over $\mathbb{R}$.

Proof of Theorem 1. Let $\mathcal{A}$ be an irreducible $n \times n$ SAP with m nonzero entries, and let $S = \{(i_{m-n+2}, j_{m-n+2}), (i_{m-n+3}, j_{m-n+3}), \ldots, (i_m, j_m)\}$ be a set of $n - 1$ positions of nonzero entries of $\mathcal{A}$ such that G_S is a tree. Let (i_1, j_1), $\ldots$, (i_{m-n+1}, j_{m-n+1}) be the remaining positions of nonzero entries of $\mathcal{A}$.

Let B be the matrix obtained from $\mathcal{A}$ by replacing the entries corresponding to the arcs of S by ± 1 according to the sign of the entry in $\mathcal{A}$, and the (i_k, j_k)-entry by $\pm X_k$ according to the sign of the entry in $\mathcal{A}$ ($k = 1, 2, \ldots, m - n + 1$).

Then the characteristic polynomial of B has the form

$$x^n - \alpha_1(X_1, \ldots, X_{m-n+1})x^{n-1} + \cdots + (-1)^n \alpha_n(X_1, \ldots, X_{m-n+1})$$

for some polynomials $\alpha_1, \alpha_2, \ldots, \alpha_{m-n+1} \in \mathbb{R}[X_1, \ldots, X_{m-n+1}]$.

Assume that $m - n + 1 < n$. Then the n α_i's are algebraically dependent over $\mathbb{R}$, and hence there exists a nonzero $p \in \mathbb{R}[X_1, \ldots, X_n]$ such that $p(\alpha_1(X_1, \ldots, X_{m-n+1}), \ldots, \alpha_n(X_1, \ldots, X_{m-n+1})) = 0$. Since A is a SAP, Lemma 2, implies that

$$\{[\alpha_1(y_1, \ldots, y_{m-n+1}), \ldots, \alpha_n(y_1, \ldots, y_{m-n+1})]^T : y_i > 0\} = \mathbb{R}^n.$$

Hence, p vanishes everywhere in $\mathbb{R}^n$. This contradicts the fact that p is nonzero.

Therefore $m - n + 1 \geq n$, and we conclude $m \geq 2n - 1$. ■

In light of Theorem 1, and known examples of SAPs, the smallest number of nonzero entries in an **irreducible** SAP of order n ($n \geq 2$) is either $2n - 1$ or $2n$. At this time there are no known examples of $n \times n$ SAPs with $2n - 1$ nonzero entries. In [DHHMMPSV06], it is shown that for $n \leq 5$ every $n \times n$ SAP has at least $2n$ nonzero entries.

Note that if $\mathcal{A}$ is an $n \times n$ irreducible sign pattern with $2n-1$ nonzero entries, S a set of positions of nonzero entries of $\mathcal{A}$ such that G_S is a tree, and the α_i's and B are defined as in the proof of Theorem 1, then a necessary condition for $\mathcal{A}$ to be a SAP is that α_1, α_2, $\ldots$, α_n are algebraically independent over $\mathbb{R}$. The algebraic independence can be efficiently checked for specific sign patterns with the aid of a computer algebra package such as Maple.

The proofs of Lemma 2 and Theorem 1 can be easily adapted[2] to prove:

Corollary 3 *If $\mathcal{A}$ is a SAP and has k irreducible components, then $\mathcal{A}$ has at least $2n - k$ nonzero entries.*

The relationship between reducibility and the spectral arbitrariness is more subtle than first sight might indicate. First note, that it need not be the case that the direct sum of SAPs is a SAP. For example, if $\mathcal{A}$ is an $n \times n$ SAP with n odd, then $\mathcal{A} \oplus \mathcal{A}$ is not a SAP because every realization of the sign pattern $\mathcal{A} \oplus \mathcal{A}$ has two real eigenvalues. Perhaps, even more surprisingly, there are examples of SAPs of the form $\mathcal{B} \oplus \mathcal{C}$ where not both $\mathcal{B}$ and $\mathcal{C}$ are SAPs [DHHMMPSV06]. Therefore, in particular, the $2n$-conjecture for reducible sign patterns would not immediately follow from the validity of the $2n$-conjecture for irreducible sign patterns.

We conclude this section with several specific problems:

Problem 1 *Determine the validity of the $2n$-conjecture for irreducible sign-patterns.*

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[2]The proof of Lemma 2 can be modified to show that there is a positive-diagonal matrix D such that the (i, j)-entry of DAD^{-1} equals $a_{ij}/|a_{ij}|$ for each (i, j) for which $i \to j$ is an arc of a prescribed depth-first search "forest" of the digraph of D.

Problem 2 *Determine the validity of the $2n$-conjecture for certain families of irreducible sign patterns (e.g. lower Hessenberg*[3]*).*

Problem 3 *Determine the validity of the $2n$-conjecture for reducible sign-patterns.*

Problem 4 *Determine necessary and sufficient conditions on the sign pattern $\mathcal{B}$ for the existence of a sign pattern $\mathcal{C}$ such that $\mathcal{B} \oplus \mathcal{C}$ is a SAP.*[4]

Problem 5 *Construct, if possible, examples of SAPs of the form $\mathcal{B} \oplus \mathcal{C}$ where neither $\mathcal{B}$ nor $\mathcal{C}$ is a SAP.*

2 NJ method

In this section we describe the most frequently used method for proving that a sign pattern is spectrally arbitrary. This technique, developed in [DJOV00], applies the Implicit Function Theorem to a polynomial function evaluated at a point corresponding to a nilpotent matrix, and thus is called the Nilpotent-Jacobi Method (or NJ-method for short).

More precisely, the set-up is as follows. Let $A = [a_{ij}]$ be an $n \times n$ nilpotent matrix, and $U = \{(i_1, j_1), \ldots, (i_n, j_n)\}$ be n positions of nonzero entries of A. Define B to be the matrix obtained from A by replacing the (i_k, j_k)-entry by the indeterminate X_k ($k = 1, 2, \ldots, n$). Then there exist polynomials $\alpha_i \in \mathbb{R}[X_1, \ldots, X_n]$ such that the characteristic polynomial of B is $x^n - \alpha_1 x^{n-1} + \cdots + (-1)^n \alpha_n$. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by

$$f(x_1, \ldots, x_n) = (\alpha_1(x_1, \ldots x_n), \ldots, \alpha_n(x_1 \ldots, x_n)).$$

The Jacobian of f at $(a_{i_1,j_1}, \ldots, a_{i_n,j_n})$ is the n by n matrix $\text{Jac}(f)$ whose (i, j)-entry is $\partial\alpha_i/\partial x_j$ evaluated at $(a_{i_1,j_1}, \ldots, a_{i_n,j_n})$.

Theorem 4 *Using the notation above, if $\text{Jac}(f)$ is nonsingular, then the sign pattern of A is a SAP.*

Proof. Suppose that $\text{Jac}(f)$ is nonsingular. Then the Implicit Function Theorem asserts that there exist open neighborhoods M of $(a_{i_1,j_1}, \ldots, a_{i_n,j_n})$ and N of $f(a_{i_1,j_1}, \ldots, a_{i_n,j_n})$ such that f maps M bijectively to N. Since

$$f(a_{i_1,j_1}, \ldots, a_{i_n,j_n}) = (0, 0, \ldots, 0)$$

it follows that for all $(c_1, c_2, \ldots, c_n)$ with each $|c_i|$ sufficiently small, there is a matrix with the same sign pattern as A with characteristic polynomial $x^n +$

[3]Note that if $\mathcal{A}$ is an n by n irreducible, lower Hessenberg sign pattern, then one may take $S = \{(1, 2), (2, 3), \ldots, (n - 1, n)\}$. With this choice, if in constructing the matrix B as in the proof of Theorem 1 we let $X_k = Y_k^{j_k - i_k}$, then resulting coefficients, $\alpha_j(Y_1, Y_2, \ldots, Y_n)$, of the characteristic polynomial of B will be homogeneous polynomials in $Y_1, \ldots, Y_n$.

[4]For example, a necessary conditions are that $\mathcal{B}$ has nilpotent realization, and that $\mathcal{B}$ has a realization with all real eigenvalues.

$c_1x^{n-1}+\cdots+c_n$. By considering all scalar multiples of such matrices, we conclude that the sign pattern of A is a SAP. ■

For example, let $n \geq 3$, and let A be the n by n matrix with 1's in each entry in its first columns, -1's in positions (n,n) and $(i,i+1)$ $(i=1,2,\ldots,n-1)$, and 0's elsewhere. Let $U=\{(i,i+1):i=1,\ldots,n-1\}$. Then A is a nilpotent matrix in $\mathcal{V}_n$, and

$$B=\begin{bmatrix} X_1 & -1 & 0 & \cdots & 0 \\ X_2 & 0 & -1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ X_{n-1} & 0 & & & -1 \\ X_n & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

It is easy to verify that $\mathrm{Jac}(f)$ is the $n\times n$ matrix with 1's on the diagonal, -1's in each subdiagonal position and 0's elsewhere. In particular, $\mathrm{Jac}(f)$ is nonsingular. Hence, by Theorem 4, $\mathcal{V}_n$ is a SAP.

A *superpattern* of the sign pattern $\mathcal{A}$ is a sign pattern obtained from $\mathcal{A}$ by replacing any collection of entries equal to 0 by nonzero signs. A slight modification of the proof of Theorem 4 leads to:

Corollary 5 *If* $\det Jac(f) \neq 0$*, then every superpattern of the sign pattern of A is a SAP.*

The NJ-method has been used to prove the following $n\times n$ matrices are SAPs with $2n$ nonzero entries.

(a) The antipodal tridiagonal pattern T_n (which is the $n\times n$ sign pattern with $-$ in positions $(1,1)$, $(2,1)$, $(3,2)$, $\ldots$, $(n,n-1)$, $+$ in positions (n,n), $(1,2)$, $(2,3)$, $\ldots$, $(n-1,n)$, and 0's elsewhere) is a SAP for $2\leq n\leq 7$ [DJOV00].

(b) T_n for $8\leq n\leq 16$ [EOV03].

(c) The $n\times n$ sign patterns $\mathcal{V}_{n,k}$ with $3\leq k+2\leq n<2k+\frac{1}{2}(\sqrt{1+8k}+3)$ which have $-$'s in positions

$$\{(j,j+1):j=1,\ldots,n-1\},\{(j,1):j=k+2,\ldots,n\},\text{ and }(n,n);$$

positive signs in positions

$$\{(j,1):j=1,\ldots,k\}\text{ and }(n,n-k);$$

and zeros elsewhere [BMOV04]

(d) $\mathcal{V}_n^{\mathrm{alt}}$ the $n\times n$ sign pattern with $+$'s in positions

$$\{(4j+1,1):0\leq j\leq\lfloor\frac{n-1}{4}\rfloor\}\text{ and }\{(n,n-(4j+1)):0\leq j\leq\lfloor\frac{n-2}{4}\rfloor\};$$

$-$'s in positions

$$\{(j, j+1) : 1 \leq j \leq n-1\};$$

$$\{(4j+3, 1) : 0 \leq j \leq \lfloor \frac{n-3}{4} \rfloor\};$$

$$\{(n, n-(4j+3)) : 0 \leq j \leq \lfloor \frac{n-4}{4} \rfloor\} \text{ and } (n,n);$$

and zeros elsewhere [BMOV04].

(e) The $n \times n$ sign patterns

$$\mathcal{D}_{n,r} = \begin{bmatrix} - & + & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ - & 0 & + & 0 & & & & \vdots \\ - & 0 & 0 & + & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ - & 0 & & & \ddots & \ddots & 0 & 0 \\ 0 & - & 0 & & & \ddots & + & 0 \\ \vdots & \ddots & \ddots & \ddots & & & 0 & + \\ 0 & \cdots & 0 & - & 0 & \cdots & 0 & + \end{bmatrix}$$

where there are r negative entries in the first column and $(2 \leq n \leq 2r)$ [CV05].

All the previously mentioned applications of the NJ-method involved the explicit construction of a nilpotent realization. In [CKSV05], the implicit function theorem is used to show the existence of a desired nilpotent realization, and the NJ-method is applied to show that the following sign patterns are SAPs, and

$$\mathcal{K}_{n,r} = \begin{bmatrix} + & - & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ + & 0 & - & 0 & & & & \vdots \\ + & 0 & 0 & - & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & & & & \ddots & - & 0 \\ + & 0 & \cdots & & & \cdots & 0 & - \\ 0 & \cdots & 0 & + & 0 & \cdots & 0 & - \end{bmatrix},$$

where the $+$ in the last row is in column r and $(2 \leq n \leq 2r)$.

The following technical result shows that the NJ-method cannot be used to establish that an $n \times n$ sign-pattern with $2n-1$ nonzero entries is a SAP.

Theorem 6 *For A, B and f defined as above, if A is irreducible of order n with $2n-1$ nonzero entries. Then* $\det \operatorname{Jac}(f) = 0$.

Proof: If each diagonal entry of A is 0, then the sign pattern of A is clearly not a SAP, and by Theorem 5, $\det \mathrm{Jac}(f) = 0$.

Now suppose that at least one diagonal entry of A is nonzero. Let $S = \{(i_k, j_k) : k = n+1, \ldots, 2n-1\}$ be the set of remaining $n-1$ positions of nonzero entries of A.

First suppose that the G_S is a spanning tree. For each t, the matrix tA is nilpotent. Since G_S is a spanning tree, a slight modification of Lemma 2 implies that there exists a positive diagonal matrix D such that $D(tA)D^{-1}$ agrees with A on the $n-1$ positions in S. It follows that every open neighborhood of $(a_{i_1,j_1}, \ldots, a_{i_n,j_n})$ contains infinitely many nilpotent matrices. Hence f is never one-to-one in a neighborhood of $(a_{i_1,j_1}, \ldots, a_{i_n,j_n})$, and therefore by the Implicit Function Theorem $\det J = 0$.

Next suppose that G_S is not a spanning tree. Let T be a set of positions of $n-1$ nonzero entries of A such that G_T is a spanning tree. By Lemma 2 there is a D (with diagonal entries products of X_i's and X_j^{-1}) such that DAD^{-1} is 1 in positions in T. Assign variables $Y_1, Y_2, \ldots, Y_n$ to nonzero entries not in the positions in T to obtain a matrix C. Note that each Y_k is a product of X_i's and X_j^{-1}'s. There exist polynomials $\beta_1, \ldots, \beta_n \in \mathbb{R}[Y_1, \ldots, Y_n]$ such that the characteristic polynomial of C is $x^n - \beta_1 x^{n-1} + \cdots + (-1)^n \beta_n$. Define $g : \mathbb{R}^n \to \mathbb{R}^n$ by $g(y_1, \ldots, y_n) = (\beta_1(y_1, \ldots, y_n), \ldots, \beta_n(y_1, \ldots, y_n))$. Let J_g be the Jacobian of g evaluated at the corresponding entries of DAD^{-1}. By the chain rule, $J(g)P = J(f)$ where P is the $n \times n$ matrix whose (i,j)-entry is $\partial x_i / \partial y_j$. Thus, by the argument in the previous case, $J(g)$ is singular, and it follows that $J(f)$ is necessarily singular. ■

Some specific problems are:

Problem 6 *Determine the values of n for which T_n is a SAP.*

Problem 7 *Characterize the irreducible, $n \times n$ SAPs with exactly $2n$ nonzero entries.*

Problem 8 *Find other methods for proving that a given sign pattern is a SAP.*[5]

3 Basic properties of SAPs

Let $\mathcal{A} = [a_{ij}]$ be an n by n sign pattern, and let D be the digraph of $\mathcal{A}$. By labeling each arc (i,j) of D by 1 if $a_{ij} = +$, and by -1 is $a_{ij} = 1$ we obtain the signed digraph of $\mathcal{A}$. A walk ω of D of length $k-1$ is a sequence $i_1, i_2, \ldots, i_k$ such $(i_\ell, i_{\ell+1})$ is an arc of D for $\ell = 1, \ldots, k-1$. If $i_1 = i_k$, then ω is a *closed* walk. The *sign* of ω is the product of the labels of its arcs. A *cycle* is a closed walk ω for which $i_1, i_2, \ldots, i_{k-1}$ are distinct. A *disjoint cycle union*, U, of D of size k is a collection of disjoint cycles of D that span k vertices. The sign of the

[5]e.g. in [BMOV04] Soules matrices where used to construct SAPs with no zero entries.

disjoint cycle union U is $(-1)^{k-\ell}p$ where p is the product of the labels of the arcs of U, and ℓ is the number of cycles of U. It is easy to verify that

$$(-1)^k \sum \text{sign}(U),$$

where the sum is over all disjoint cycle unions of size $n-k$ is equal to the coefficient c_k of x^k in the characteristic polynomial of a, where A is the $(0,1,-1)$ matrix in $\mathcal{A}$. Note that if $\mathcal{A}$ is a SAP, then c_k must have at least one positive term and one negative term. Hence, we have the following.

Proposition 7 *If $\mathcal{A}$ is an $n \times n$ SAP, then for each k with $1 \leq k \leq n$, the signed digraph of $\mathcal{A}$ has a positive disjoint cycle union of size k, and a negative disjoint cycle union of size k. In particular, if $\mathcal{A}$ is a SAP, then at least one diagonal entry of $\mathcal{A}$ is $+$ and at least one negative diagonal entry of $\mathcal{A}$ is $-$.*

By the Newton-Girard formulas [W06], $\mathcal{A}$ is a SAP if and only if for each n-tuple $(r_1, r_2, \ldots, r_n)$ of real numbers there exists a realization $A \in \mathcal{A}$ such that $\text{tr}(A^i) = r_i$ $(i = 1, \ldots, n)$. This implies the following.

Proposition 8 *If $\mathcal{A}$ is an $n \times n$ SAP with signed digraph D, then for each even integer k, D contains a negative closed walk of length k. In particular, D contains a negative cycle of length 2.*

Now assume that $\mathcal{A}$ is an irreducible, $n \times n$ sign pattern with $2n-1$ nonzero entries. In this case there are some additional necessary conditions for $\mathcal{A}$ to be a SAP. Let S be a set of positions of nonzero positions of $\mathcal{A}$ such that G_S is a tree, and, as previously, define B to be the matrix obtained from $\mathcal{A}$ by replacing the entries in nonzero positions not in S by $\pm X_1, \ldots, \pm X_n$ according to the sign of the entries of $\mathcal{A}$. There exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}[X_1, \ldots, X_n]$ such that the characteristic polynomial of B is $x^n - \alpha_1 x^{n-1} + \cdots + (-1)^n \alpha_n$. For elements $p_i \in \mathbb{R}[X_1, \ldots, X_n]$ let $I(p_1, \ldots, p_k)$ denote the ideal of $\mathbb{R}[X_1, \ldots, X_n]$ generated by $p_1, \ldots, p_k$, and let $\sqrt{I(p_1, \ldots, p_k)}$ denote the radical of $I(p_1, \ldots, p_k)$, that is, $\sqrt{I(p_1, \ldots, p_k)} = \{p : \text{ there exists a positive integer } \ell \text{ such that } p^\ell \in I(p_1, \ldots, p_k)\}$.

Proposition 9 *Let $\mathcal{A}$ be an irreducible $n \times n$ sign pattern with $2n-1$ nonzero entries, and define the α_i's and B as above. If at least one of the following conditions is satisfied, then $\mathcal{A}$ is not a SAP.*

(a) There exists a j such that X_j is on the main diagonal of B, and a nonzero polynomial $p \in \mathbb{R}[X]$ such that $p(X_j) \in \sqrt{I(\alpha_1, \alpha_2, \ldots, \alpha_n)}$.

(b) There exists a j and a nonzero polynomial $q \in \mathbb{R}[X]$ such that $\alpha_j - q \in \sqrt{I(\aleph)}$, where $\aleph$ is any collection of that α_i's that doesn't contain α_j.

(c) The variety generated by $\alpha_i(X_1, \ldots, X_n) - X_{n+i}$ has dimension less than n.[6]

[6]Note: computer algebra packages such as Maple allow one effectively compute the dimension of a variety.

Proof. To prove (a), assume that such a j and p exist. Suppose that the assignment $X_1 = x_1, \ldots X_n = x_n$ produces a nilpotent matrix B. Since $p \in \sqrt{I(\alpha_1, \alpha_2, \ldots \alpha_n)}$, there is an integer k, and polynomials q_i such that $p^k = \sum_{\alpha_i q_i}$. Evaluating at $X_1 = x_1, \ldots, X_n = x_n$ results in $p^k(x_1) = 0$. Hence, $p(x_1) = 0$. Therefore, there are only finitely many values of X_j for which the evaluation of B is nilpotent. However, by the argument in Theorem 6, we see that if B allows nilpotency, then in every neighborhood of the nilpotent realization there are an infinite number of nilpotents. This contradicts the fact that there are just finitely many values of X_j. This proves (a).

To prove (b), suppose that there exists such a j, q, and $\aleph$. Then there exist polynomials q_i such that $\alpha_j - q = \sum_{\alpha_i \in \aleph} q_i$. Thus for each assignment $X_1 = x_1, \ldots, X_n = x_n$ such that $\alpha_i(x_1, \ldots, x_n) = 0$ for $\alpha_i \in \aleph$, we have that $\alpha_j(x_1, x_2, \ldots, x_n) = q(x_j)$ As the range of $q(X_j)$ for $X_j \geq 0$ is not all of $\mathbb{R}$, there is a real number r with the property that there is no way to assign real numbers to the X_i so that $\alpha_i = 0$ for $i \in \aleph$ and $\alpha_j = r$. Hence, $\mathcal{A}$ is not a SAP.

Statement (c) is equivalent to the fact that $\alpha_1, \ldots, \alpha_n$ are algebraically dependent over $\mathbb{R}$ [CLO96], and hence as in the proof of Theorem 1, $\mathcal{A}$ is not a SAP.

4 Star patterns

The notion of SAPs can also be generalized to star patterns. The stars[7] of $\mathbb{R}$, are $\{0\}$, $(0, \infty)$, $[0, \infty)$, $(-\infty, 0)$, $(-\infty, 0]$, $\mathbb{R} \setminus \{0\}$, and $\mathbb{R}$. A *star pattern* is an n by n matrix $\mathfrak{C} = [\mathfrak{c}_{ij}]$ each whose entries is a star of $\mathbb{R}$. Note that sign patterns correspond to star patterns where each entry belongs to $\{\{0\}, (0, \infty), (-\infty, 0)\}$ and zero-nonzero patterns correspond to star patterns where each entry belongs to $\{\{0\}, \mathbb{R} \setminus \{0\}\}$. A *realization* of $\mathfrak{C}$ is a matrix each of whose entries belongs to the star specified by the corresponding entry of $\mathfrak{C}$. The star pattern $\mathfrak{C}$ is *spectrally arbitrary* provided every real, monic polynomial $r(x)$ of degree n is the characteristic polynomial of at least one realization of $\mathfrak{C}$.

There are irreducible, spectrally arbitrary $n \times n$ star patterns with $2n - 1$ nonzero entries. For example, every real polynomial $r(x) = x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_n$ is the characteristic polynomial of its companion matrix

$$\begin{bmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_{n-1} & 0 & & 0 & 1 \\ c_n & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence, the $n \times n$ star pattern $\mathfrak{K}_n$ with $\mathbb{R}$ in each entry in its first column, and in positions $(i, i + 1)$ for $i = 1, 2, \ldots, n - 1$, and 0 in all other entries, is spectrally arbitrary. Moreover, the direct sum of k $\mathfrak{K}_4$'s has order $n = 4k$, $(7/4)n$

[7]Here, a star is a set which is closed multiplication by the positive real numbers.

nonzeros, and is spectrally arbitrary. Thus, the analog of the $2n$-conjecture for star patterns is blatantly false.

Theorem 1 can be extended to star patterns. We first establish a result similar to Lemma 2. Let $\mathfrak{C}$ be an $n \times n$ star pattern, and let be a subset $S = \{(i_1, i_2), \ldots, (i_k, j_k)\}$ of positions whose entries in $\mathfrak{C}$ are not equal to $\{0\}$. Define G_S to be graph with vertices $1, 2, \ldots, n$ and edges $\{i_\ell, j_\ell\}$ $\ell = 1, \ldots, k$.

Lemma 10 *Let $\mathfrak{C}$ be an irreducible star pattern, and S a subset of positions of entries in $\mathfrak{C}$ that are not equal to $\{0\}$ such that G_S is a tree. For each matrix $C \in \mathfrak{C}$ there exists a positive-diagonal matrix D such that the (i, j)-entry of DCD^{-1} is 0 or $a_{ij}/|a_{ij}|$ for each $(i, j) \in S$.*

Proof: Without loss of generality we may assume that the edges of G_S are (i_1, j_1), ..., (i_{n-1}, j_{n-1}) and that exactly one of i_k and j_k is in $\{1, 2, \ldots, k\}$ for $k = 1, 2, \ldots, n-1$. Set $d_1 = 1$, and recursively define $d_2, \ldots, d_n$ as follows. If $i_k \notin \{1, 2, \ldots, k\}$ and $a_{i_k, j_k} \neq \{0\}$, then set $d_{i_k} = d_{j_k}/|a_{i_k, j_k}|$. If $i_k \notin \{1, 2, \ldots, k\}$ and $a_{i_k, j_k} = \{0\}$, then set $d_{i_k} = 1$. If $j_k \notin \{1, 2, \ldots, k\}$ and $a_{i_k, j_k} \neq \{0\}$, then set $d_{j_k} = d_{i_k}|a_{i_k, j_k}|$. If $j_k \notin \{1, 2, \ldots, k\}$ and $a_{i_k, j_k} = \{0\}$, then set $d_{j_k} = 1$. It is now easy to verify that $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is a positive-diagonal matrix with the desired property. ■

Theorem 11 *If $\mathfrak{C} = [\mathfrak{c}_{ij}]$ is an irreducible, $n \times n$ star pattern, then $\mathfrak{C}$ has at least $2n - 1$ nonzero entries.*

Proof: Let m be the number of nonzero entries of $\mathfrak{C}$, and let T be a subset of positions whose entries in $\mathfrak{C}$ are not equal to $\{0\}$, and G_T is a tree.

Let (i_k, j_k) $(k = 1, 2, \ldots, m-n+1)$ be the positions of the non-$\{0\}$ entries of $\mathfrak{C}$ that are not in T, and let $X_1, \ldots, X_{m-n+1}$ be distinct indeterminates. Let S be the set of arcs of T whose corresponding entry (which is a star of $\mathbb{R}$) in $\mathfrak{C}$ contains 0. For each subset R of S let B_R be the matrix obtained from $\mathcal{A}$ by replacing the entries corresponding to the arcs of S by 0, the entries corresponding to arcs of T not in S by 1, and the (i_k, j_k)-entry by X_k $(k = 1, 2, \ldots, m - n + 1)$. Then the characteristic polynomial of B_R has the form

$$x^n + \alpha_1^R(X_1, \ldots, X_{m-n+1})x^{n-1} + \cdots + \alpha_n^R(X_1, \ldots, X_{m-n+1})$$

for some polynomials $\alpha_1^R, \alpha_2^R, \ldots, \alpha_n^R \in \mathbb{R}[X_1, \ldots, X_{m-n+1}]$.

Assume that $m - n + 1 < n$. Then, for each R, the n α_i^R's are algebraically dependent over $\mathbb{R}$, and hence there exists a nonzero polynomial p^R with real coefficients such that

$$p^R(\alpha_1(X_1, \ldots, X_{m-n+1}), \ldots, \alpha_n(X_1, \ldots, X_{m-n+1})) = 0. \quad (1)$$

Since $\mathfrak{C}$ is a SAP, Lemma 10 implies that

$$\mathbb{R}^n = \cup_{R \subseteq S}\{[\alpha^R(x_1, \ldots, x_{m-n+1}), \ldots, \alpha_n^R(x_1, \ldots, x_{m-n+1})]^T : x_k \in \mathfrak{c}_{i_k, j_k}\} \quad (2)$$

Let $p(X_1, \ldots, X_{m-n+1})$ be the product of all of the p^R's where R runs over the subsets of S. Then (1) and (2) imply that $p(y_1, \ldots, y_n) = 0$ for all $(y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$. This leads to the contradiction that p is the zero polynomial.

Hence $m - n + 1 \geq n$, and we conclude that $m \geq 2n - 1$. ■

A similar technique (with G_T a spanning forest, rather than a tree) establishes the following:

Corollary 12 *If $\mathfrak{C}$ is an n by n, spectrally arbitrary star pattern with k irreducible components, then $\mathfrak{C}$ has at least $2n - k$ nonzero entries.*

Some specific problems concerning spectrally arbitrary star patterns are the following.

Problem 9 *Characterize the $n \times n$, irreducible and spectrally arbitrary star patterns with exactly $2n - 1$ non-$\{0\}$ entries.*

Problem 10 *For each integer $n \geq 2$, determine the minimum number of nonzero entries in an $n \times n$ spectrally arbitrary star pattern.*

Problem 11 *For each integer $n \geq 2$, determine the minimum number of $\mathfrak{c}_{ij}$ such that $0 \in \mathfrak{c}_{ij} \neq \{0\}$, where $\mathfrak{C} = [\mathfrak{c}_{ij}]$ is an n by n, irreducible and spectrally arbitrary star pattern with $2n - 1$ nonzero entries.*

Problem 12 *Determine whether or not there exists an irreducible, $n \times n$, irreducible and spectrally arbitrary star pattern with $2n - 1$ nonzero entries and at least n of which do not contain 0 as an element.*

Problem 13 *Determine whether or not there exists an irreducible, $n \times n$ spectrally arbitrary, zero-nonzero pattern with $2n - 1$ nonzero entries.*

5 Positive Nullstellensatz

In this section we describe how the problem of determining whether or not a given polynomial $r(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ is the characteristic polynomial of a realization of a given sign pattern $\mathcal{A}$ can be phrased as a problem about semialgebraic sets. The goal here is to introduce a potentially powerful algebraic tool into the study of SAPs. A *primal semialgebraic problem* is an existence question of the form:

> Given $f_i, h_j \in \mathbb{R}^n$ $(i = 1, 2 \ldots, m; j = 1, 2, \ldots, p)$, does there exists $x \in \mathbb{R}^n$ such that
>
> $$\begin{aligned} f_i(x) &\geq 0 \text{ for all } i = 1, \ldots, m \\ g_j(x) &\neq 0 \text{ for all } j = 1, \ldots, p \text{ and} \\ h_j(x) &= 0 \text{ for all } j = 1, \ldots, q? \end{aligned} \tag{3}$$

The problem of determining if a given sign pattern $\mathcal{A} = [a_{ij}]$ has a realization with characteristic polynomial $r(x) = x^n + c_1x^{n-1} + \cdots + c_n$ can be phrased as a semialgebraic problem as follows.

Assume that $\mathcal{A} = [a_{ij}]$ has m nonzero entries, say in positions (i_1, j_1), ..., (i_m, j_m). Let B be the matrix obtained from $\mathcal{A}$ by replacing its (i_k, j_k)-entry by the indeterminate X_k if $a_{i_k,j_k} = +$ and by $-X_k$ otherwise ($k = 1, 2, \ldots, m$). Then there exist polynomials $\alpha_1, \ldots, \alpha_n \in \mathbb{R}[X_1, \ldots, X_m]$ such that the characteristic polynomial of B is

$$x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n.$$

We claim that $r(x)$ is the characteristic polynomial of a realization of $\mathcal{A}$ if and only if the primal problem

$$\begin{aligned} \alpha_i - c_i = 0 \quad & (i = 1, 2, \ldots, n) \\ X_i \geq 0 \quad & (i = 1, 2, \ldots, n) \\ X_i \neq 0 \quad & (i = 1, 2, \ldots, n) \end{aligned}$$

has a solution.[8] To argue this first suppos that $r(x)$ is the characteristic polynomial of $A \in \mathcal{A}$. Let x_i equal the value of the entry of A corresponding to X_i. Then $c_i = \alpha_i(x_1, \ldots, x_m)$, and $x_i > 0$. Thus the claimed primal problem has a solution. Conversely, suppose that $X_i = x_i$ ($i = 1, 2, \ldots, m$) is a solution to the primal problem. Then $x_i > 0$, $\alpha_i(x_1, \ldots, x_n) = c_i$. Thus, if we replace X_i in B by x_i we obtain the a realization of $\mathcal{A}$ with characteristic polynomial $r(x)$.

We now describe necessary and sufficient conditions (first proven in [S72], see also [BCR98]) for the existence of a solution to (3). First we need a few definitions. Let P be the subset of all polynomials in $\mathbb{R}[X_1, X_2, \ldots, X_n]$ that can be expressed as the sum of squares of elements of $\mathbb{R}[X_1, X_2, \ldots, X_n]$. Let S be a subset of $\mathbb{R}[X_1, X_2, \ldots, X_n]$. We let $I(S)$ denote the ideal of $\mathbb{R}[X_1, \ldots, X_n]$ generated by the set S, $M(S)$ denote the multiplicative monoid generated S of $\mathbb{R}[X_1, \ldots, X_n]$ (that is, $M(S)$ is the set of all finite products of elements of S), and Cone(S) denote the cone generated by S (which is the set of finite sums of the form $p + \sum_{i=1}^{r} q_i b_i$ such that $p, q_1, \ldots, q_r \in P$ and $b_1, b_2, \ldots, b_r \in M(S)$. Note that if $S = \emptyset$, then Cone(S) $= P$, $M(S) = \{1\}$ (from the empty product), and $I(S) = \{0\}$.

Theorem 13 *The basic primal semialgebraic problem* (3) *has no solution if and only if there exist* $f \in \mathit{Cone}(f_1, \ldots, f_m)$, $g \in M(g_1, g_2, \ldots, g_p)$, *and* $h \in I(h_1, h_2, \ldots, h_q)$ *such that* $f + g^2 + h = 0$.

Note that the only if direction of Theorem 13 is easily verified: if there is a solution $(x_1, \ldots, x_n)$ to the primal semialgebraic problem (3), then for $f \in$ Cone$(f_1, \ldots, f_m)$, $g \in M(g_1, \ldots, g_p)$ and $h \in I(h_1, \ldots, h_n)$ we have $f(x_1, \ldots, x_n) \geq 0$, $g(x_1, x_2, \ldots, x_n) > 0$, and $h(x_1, \ldots, x_n) = 0$, and hence $f + g^2 + h \neq 0$.

Note that if $S = \{X_1, \ldots, X_n\}$, then every element of $M(S)$ other than 1 belongs to Cone(S). Hence, we have the following:

[8] Note via the use of diagonal scalings, one can reduce the number of variables to $m-(n+1)$.

Corollary 14 *Let $\mathcal{A}$ be an $n \times n$ sign pattern, and define B, $\alpha_1, \ldots, \alpha_n$ as above. Then no realization $A \in \mathcal{A}$ has characteristic polynomial $x^n + c_1 x_1 + \cdots + c_n$ if and only if there exist $f \in Cone(X_1, X_2, \ldots, X_n)$, and $h \in I(\alpha_1 - c_1, \ldots, \alpha_n - c_n)$ such that $f + h = -1$.*

For example, consider a matrix B of the form

$$\begin{bmatrix} X_1 & -1 & 0 \\ X_2 & -X_3 & 1 \\ X_4 & X_5 & -X_6 \end{bmatrix}$$

The characteristic polynomial of B is $x^3 - \alpha_1 X^2 + \alpha_2 X - \alpha_3$ where

$$\begin{aligned} \alpha_1 &= X_1 - X_3 - X_6 \\ \alpha_2 &= -X_1X_3 + X_2 - X_1X_6 + X_3X_6 + X_5 \\ \alpha_3 &= -X_1X_3X_6 + X_4 + X_2X_6 + X_1X_5. \end{aligned}$$

It is easy to verify that

$$-\alpha_3 + X_6\alpha_2 + (X_6^2 + X_5)\alpha_1 + (X_4 + X_3X_5 + X_6^3) = 0.$$

Since $-\alpha_3 - X_6\alpha_2 + (X_6^2 + X_5)\alpha_1 \in I(\alpha_1, \alpha_2, \alpha_3)$ and $(X_4 + X_3X_5 + X_6^3) \in \text{Cone}(X_1, \ldots, X_6)$, we conclude that

$$\begin{bmatrix} + & - & 0 \\ + & - & + \\ + & + & - \end{bmatrix}$$

is not a SAP.

Thus, we see that the only if portion of Corollary 14 gives a way to certify that a sign pattern $\mathcal{A}$ is not a SAP; namely, find real numbers $c_1, \ldots, c_n$, $f \in \text{Cone}(X_1, X_2, \ldots, X_n)$, $h \in I(\alpha_1 - c_1, \ldots, \alpha_n - c_n)$ with $f + h = -1$. More importantly, Corollary 14 says that such certifications are the only ways to show that a sign pattern in not a SAP.

Problem 14 *Can the problem of determining whether or not a sign pattern is a SAP be phrased as a basic semialgebraic problem?*

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