NUMERICAL RENORMALIZATION GROUP ALGORITHMS FOR SELF-SIMILAR ASYMPTOTICS OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We systematically study a numerical procedure that reveals the asymptotically self-similar dynamics of solutions of partial differential equations (PDEs). This procedure, based on the renormalization group (RG) theory for PDEs, appeared initially in a conference proceedings [11]. A numerical version of the RG method, dubbed nRG, rescales the temporal and spatial variables in each iteration and drives the solutions to a fixed point exponentially fast, which corresponds to the self-similar dynamics of the equations. In this paper, we carefully examine and validate this class of algorithms by comparing the numerical solutions with either exact or asymptotic solutions of model equations found in the literature. The other contribution of the current paper is that we present several examples to demonstrate that this class of nRG algorithms can be applied to a wide range of PDEs to shed light on long time self-similar dynamics of certain physical systems modeled by PDEs.

keywords: Renormalization group theory, numerical renormalization group, asymptotic limit, self-similar dynamics, critical exponents, logarithmic corrections, marginal perturbations

1. Introduction

Solutions of many partial differential equations (PDEs) in continuum mechanics exhibit self-similar dynamics in the asymptotic limit of large times, namely

\[ u(x,t) \sim \frac{A}{t^\alpha} \phi\left( \frac{Bx}{t^\beta} \right), \quad \text{as } t \to \infty. \]

(1)

This particular behavior indicates that in the long time asymptotic limit, the dynamics of the solution are controlled by two factors, the decay in the magnitude of \( u \) and the spread of its spatial distribution, whose profile is given by the function \( \phi \). The constants \( A \) and \( B \) in Eq. (1) are usually related to the initial conditions through some conservation laws of the PDE under study.

Physicists used the renormalization group to obtain the scaling regime of a variety of physical models in equilibrium statistical mechanics and quantum field theory [27, 29, 39, 44, 49, 50]. In the early 1990s, Goldenfeld, Oono, and collaborators developed a perturbative renormalization group method for PDEs and applied it to the study of a number of large-time asymptotic problems [18, 29–31, 39]. In addition, a number of papers have obtained asymptotic scaling exponents by analytic renormalization group methods [15–17, 38]. At nearly the same time, a nonperturbative RG approach was introduced by Bricmont et al. [12, 14], and was applied to the study of nonlinear dispersive and dissipative wave equations and thermal-diffusive combustion system [8, 13, 40].

Also in the 1990s, Chen and Goldenfeld proposed a numerical RG (nRG) calculation for similarity solutions of the porous medium (Barenblatt) equation and traveling waves [19]. Their numerical procedure inspired mesh renormalization methods for studying focusing problems arising in porous medium flow [4, 7]. Inspired by the numerical approach of Chen and Goldenfeld and the nonperturbative RG approach of Bricmont and Kupianinen, Braga et al. [11] introduced a class of nRG algorithms in a short conference paper that allow them to systematically search for the critical exponents and the hidden decay in asymptotically self-similar dynamics through repeated scalings in time and space.

In this paper, we carefully examine and validate the nRG algorithms of Braga et al. [11] by comparing the numerical solutions of the nRG algorithms with either exact or asymptotic solutions of model equations in the literature. We show that the self-similar dynamics captured by the nRG algorithms agree with the theoretical results for scalar equations, as well as for systems of equations. Furthermore, a novel
contribution of this paper is that we demonstrate that this nRG procedure can be used to shed light on the behavior of self-similar asymptotics of certain physical models, such as the nonlinear diffusion-absorption model, that are difficult to analyze, both numerically and analytically.

It is worth noting that the procedure introduced by Braga et al. [10] for studying a nonlinear diffusion equation with periodic coefficients is similar in spirit to one of the nRG algorithms studied in this paper. Numerical procedures based on rescaling the solutions and the time and spatial variables have previously been developed and used to study solutions of PDEs that blow up in finite time [6,21,22,24,33,36,42,51]. Such procedures exploit the known self-similar structure of the solutions under study to determine the appropriate rescalings.

The nRG algorithms in this paper, however, are unique in exploiting fixed points by performing successive iterations of a discrete RG transformation in space and time that drive the system towards a fixed point; however, the current implementation of the algorithms in this paper is not suitable for studying blow-up problems. A numerical procedure based on the renormalization group theory to compute the spatial profile and blow-up time for self-similar behavior was proposed by Isaia [35]. This version of nRG algorithm uses Berger-Kohn’s time stepping [6], along with another approach that computes the blow-up rate.

Finally, to aid the reader, we outline the contents of the remainder of this paper. In Section 2, using Burgers equation, we explain the fundamental idea of the nRG algorithm and outline its steps. We show that the algorithm allows the computation of the scaling exponents, in time and space, of the asymptotic self-similar behavior of solutions. In addition, we demonstrate its ability to render explicit the relative importance of the distinct effects represented by the terms in the equation in the long time behavior of solutions; we remark that this relative importance may change, depending on the class of initial conditions considered.

In Section 3, we use the phenomenon of dispersive shock waves for the Korteweg-de Vries equation to show that the algorithm can be used as an efficient time integrator to investigate the intermediate asymptotic behavior of solutions. In Section 4, we study a class of nonlinear diffusion-absorption models. A conjecture on the existence of a critical exponent of the nonlinear absorption term is made for problems with discontinuous diffusivities. We also present a marginal case, for which the phenomenon of anomalous decay is observed, as motivation for the next section. Finally, in Sections 5, 6 and 7, we present a modified nRG algorithm and illustrate its ability to detect and capture the hidden logarithmic decay of solutions to a nonlinear system of reaction-diffusion equations modeling cubic autocatalytic chemical reactions.

2. Scaling Transformations for Burgers Equation

The main idea behind the nRG method is to obtain the long time asymptotics of solutions of PDEs iteratively. First, we integrate (solve) the PDE over a finite time-interval with fixed length. Then, we rescale the solution so as to produce a new problem similar to the original one and, finally, we iterate. The rescaling usually renormalizes the terms of the PDE. Once a proper scaling has been found, upon iteration this tends to stabilize to a fixed equation, whose scale invariant solution gives the long time behavior of the solutions of the original problem. To explain this idea, we use Burgers equation as an example to illustrate the scaling transformation procedure of the nRG algorithm. We then compare the asymptotic solutions obtained by the nRG algorithm with the exact solutions in the asymptotic regime to demonstrate the robust nature of our algorithm.

Burgers equation, with initial condition at \( t = 1 \), is written as

\[
\begin{align*}
\partial_t u + u \partial_x u &= \nu \partial_x^2 u, \quad x \in \mathbb{R}, \quad t > 1, \\
\text{I. C. :} \quad u(x, 1) &= f(x), \quad x \in \mathbb{R}.
\end{align*}
\]

(2)

Here \( \nu > 0 \) is the viscosity; \( f(x) \) is a smooth and rapidly decaying initial condition. Let the time and space variables be scaled as follows,

\[
t = \tilde{L} \tilde{t}, \quad x = L^\beta \tilde{x},
\]

(3)
with the parameter $L > 1$ fixed, $\beta > 0$, $\tilde{t}$ and $\tilde{x}$ denoting the new variables. Suppose the solution $u$ of the initial value problem (IVP) (2) is scaled similarly:

$$u_L(\tilde{x}, \tilde{t}) = L^\alpha u(x, t) = L^\alpha u(L^\beta \tilde{x}, L\tilde{t}),$$  

where $\alpha > 0$. With these scalings, each of the terms in Burgers equation scales accordingly:

$$u_t = L^{-(\alpha + 1)} \frac{\partial}{\partial t} u_L(\tilde{x}, \tilde{t}), \quad u_x = L^{-(\alpha + \beta)} \frac{\partial}{\partial \tilde{x}} u_L(\tilde{x}, \tilde{t}), \quad u_{xx} = L^{-(\alpha + 2\beta)} \frac{\partial^2}{\partial \tilde{x}^2} u_L(\tilde{x}, \tilde{t}).$$  

Substituting (5) into (2) yields the renormalized initial value problem satisfied by $u_L$:

$$\frac{\partial}{\partial \tilde{t}} u_L + L^{1-\alpha-\beta} u_L \frac{\partial}{\partial \tilde{x}} u_L = \nu L^{1-2\beta} \frac{\partial^2}{\partial \tilde{x}^2} u_L, \quad \tilde{t} > 1,$$

$$I. C. : \quad u_L(\tilde{x}, 1) = \tilde{f}(\tilde{x}),$$

where $\tilde{f}(\tilde{x}) = L^\alpha u(L^\beta \tilde{x}, L)$.  

2.1. Sequence of Scaling Transformations. Performing a sequence of scalings (iterations) with a fixed $L > 1$ and sequences of scaling exponents $\{\alpha_n\}$ and $\{\beta_n\}$, we define a sequence of scaled functions $\{u_n\}$ by rewriting Eq. (4) (dropping the $\sim$ in $\tilde{x}$ and $\tilde{t}$) as

$$u_n(x, t) = L^{\alpha_n} u_{n-1}(L^{\beta_n} x, L t),$$

with $u_0 = u$, the solution of the original IVP (2). A simple, but important, calculation using (7) demonstrates the preservation of the semigroup property and establishes an important relationship between $u_n$ and $u$:

$$u_n(x, t) = L^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} u(L^{\beta_1} + L^{\beta_2} + \cdots + L^{\beta_n} x, L^n t)$$

$$= L^{n\bar{\alpha}_n} u(L^{n\bar{\beta}_n} x, L^n t),$$

where $\bar{\alpha}_n = (\alpha_1 + \alpha_2 + \cdots + \alpha_n)/n$ and $\bar{\beta}_n = (\beta_1 + \beta_2 + \cdots + \beta_n)/n$. In particular, Eq. (8) shows how $u_n$ in the time interval $t \in [1, L]$ is related to $u$ in the time interval $t \in [L^n, L^{n+1}]$. Since in each iteration the scaling of the PDE, as performed in Eq. (6), is applied to the previous scaled equation, the solution $u_n$ of the $n^{th}$ iteration is the solution of the following renormalized initial value problem:

$$\frac{\partial}{\partial \tilde{t}} u_n + L^{n(1-\bar{\alpha}_n-\bar{\beta}_n)} u_n \frac{\partial}{\partial \tilde{x}} u_n = \nu L^{n(1-2\bar{\beta}_n)} \frac{\partial^2}{\partial \tilde{x}^2} u_n, \quad t > 1,$$

$$I. C. : \quad u_n(x, 1) = f_n(x),$$

where $f_n(x) = L^{\alpha_n} u_{n-1}(L^{\beta_n} x, L)$, with $f_0 = f$, the initial condition of IVP (2). From (8) and the definition of $f_n$, we deduce that

$$u(x, L^n) = \frac{A_n}{L^{n\alpha_n}} f_n \left( B_n \frac{x}{L^{n\beta_n}} \right),$$

with

$$A_n = L^{n(\alpha_n - \bar{\alpha}_n)}, \quad B_n = L^{n(\beta_n - \bar{\beta}_n)}.$$

In other words, the preservation of the semigroup property allows the long time behavior for the problem under study, which is encoded in $u(x, L^n)$, to be transferred via the iteration to the rescaled (and easy to compute) construction $\frac{A_n}{L^{n\alpha_n}} f_n \left( B_n \frac{x}{L^{n\beta_n}} \right)$. Therefore, if as $n \to \infty$ the limits $\alpha_n \to \alpha$, $\beta_n \to \beta$, $f_n \to \phi$, $A_n \to A$ and $B_n \to B$ are attained, it is reasonable to expect that

$$L^{n\alpha} u(L^{n\beta} x, L^n) = \frac{A_n}{L^{n(\alpha_n - \alpha)}} f_n \left( B_n \frac{x}{L^n(\beta_n - \beta)} \right) \to A \phi(Bx), \quad n \to \infty.$$  

This limit, in turn, gives the long time, self-similar asymptotics of $u$, Eq. (1), by letting $t = L^n$.  

Next, we introduce the numerical RG algorithm. Then, we use it to explore the well known self-similar asymptotic decay of solutions of Burgers equation with initial data in two distinct classes: (i) nonnegative functions with compact support; (ii) functions with zero mass and compact support. We do so to exhibit the precision of the nRG method to determine asymptotic scaling exponents and profile functions iteratively, as limits of computed sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{f_n\}$ of scaling exponents and initial
conditions, as well as its capability to reveal the dominant effects in the asymptotic regime. The latter are suggested by the way the terms in the equation are renormalized upon iteration, see (9).

2.2. The nRG procedure. We describe the nRG algorithm in Algorithm 1.

Let us pause to discuss the strategy to determine the scaling exponents, which define the scaling transformations in the nRG procedure, in steps 2 and 3.

The rationale behind the manner we compute the sequence \{α_n\} in step 2 is the self-similar asymptotic behavior we want to explore. In the self-similar regime, the solution \(u\) decays like \(t^{-\alpha}\) (for some \(\alpha\)), see (1). By virtue of the relation between \(u_0\) and \(u\) (see Eq. (8)), the decay of \(u_n\) over the time interval \([1, L]\) is equivalent to the decay of \(u\) over the time interval \([L^n, L^{n+1}]\). Therefore, by measuring the decay of \(u_n\) in terms of the exponent \(\alpha_{n+1}\), as we do in step 2, the limit \(L^{\alpha_{n+1}} \rightarrow L^\alpha\) as \(n \rightarrow \infty\) should be attained, i.e., \(\alpha_{n+1} \rightarrow \alpha\).

The determination of the exponents \(\beta_n\) is problem dependent. It usually involves a scaling relation between the exponents \(\alpha_n\) and \(\beta_n\), which is chosen so that certain (a priori selected) terms of the differential operator in consideration remain invariant under the scaling of step 4. Once a proper scaling relation has been found for a particular problem, the terms of the differential operator are divided into two types: neutral or irrelevant, according to whether their magnitude is unchanged (asymptotically) or decreases with each RG iteration. The irrelevant terms iterate to zero and the dynamics at large times is then controlled by the neutral terms.

To illustrate this point, consider Burgers equation and its renormalization (9). If \(\beta_n\) is chosen to be 1/2, the linear operator \(\partial_t u - \nu \partial_x^2 u\) remains unchanged under the scaling in step 4. So this choice focuses the attention on the dynamics of the equation \(\partial_t u = \nu \partial_x^2 u\) and, as such, is suitable for the investigation of its irrelevant and neutral perturbations. On the other hand, if \(\beta_n\) is chosen to satisfy the scaling relation 1 - \(\alpha_n - \beta_n\) = 0 (i.e., \(\beta_n = 1 - \alpha_n\)), it is the quasilinear operator \(\partial_t u + u \partial_x u\) that remains unchanged. Therefore this is the appropriate scaling if we want to study irrelevant and neutral perturbations to the quasilinear equation \(\partial_t u + u \partial_x u = 0\).

2.3. Numerical experiments. We now discuss the numerical implementation of the nRG procedure. For the numerical solution of the scaled equation (9) in each iteration of the nRG algorithm, any appropriate method can be employed. We chose a simple explicit finite difference method that combines the first-order Euler method for the temporal discretization with centered differences for the spatial derivatives. Given the stability constraints, the resulting method is second order accurate.

If we set
\[
\kappa_n = L^n(1-\bar{\alpha}_n-\bar{\beta}_n), \quad \nu_n = \nu L^{n(1-2\bar{\beta}_n)}, \quad v = u_n, \quad (13)
\]
the fully discretized equation (9) at the \(i\)th spatial mesh point and time \(t^{i+1} = (j + 1)\Delta t\) is then
\[
v_i^{j+1} = v_i^j - \Delta t \frac{\kappa_n}{2} \left( \frac{(v_{i+1}^j)^2 - (v_{i-1}^j)^2}{2\Delta x_n} \right) + \Delta t \nu_n \left( \frac{v_{i-1}^j - 2v_i^j + v_{i+1}^j}{\Delta x_n^2} \right), \quad (14)
\]
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where $\Delta x_n$ is the spatial mesh size for the $n^{th}$ iteration.

The implementation of the spatial scaling in step 4 of the algorithm requires consideration. Denoting the location of the $i^{th}$ spatial node on a uniform mesh in the $n^{th}$ iteration by $(x_i)_n = i\Delta x_n$, $i = 0, \pm 1, \pm 2, \ldots$, we deduce from equation (7) that

$$u_n(i\Delta x_n, t) = L^{\alpha_n} u_{n-1} \left( L^{\beta_n} i\Delta x_n, Lt \right) = L^{\alpha_n} u_{n-1} (i\Delta x_{n-1}, Lt).$$

(15)

It follows from this that

$$\Delta x_n = L^{-\beta_n} \Delta x_{n-1}.$$  

(16)

Since $L > 1$, if $\beta_n > 0$ for all $n$, then $\Delta x_n < \Delta x_{n-1} < \cdots < \Delta x_0$, and for sufficiently large $n$, $\Delta x_n \ll \Delta x_0$. If the same $\Delta t$ is used in all iterations, this could eventually result in numerical instability, in case an explicit time integrator, such as the Euler method adopted in this paper, is used. Conversely, if $\Delta t$ is decreased accordingly to obey the stability requirement, the time integration eventually becomes too costly for the nRG algorithm. A possible remedy for this situation is to always use the same $\Delta t$, i.e., $\Delta x_n = \Delta x_{n-1} = \cdots = \Delta x_0 = \Delta x$, and to represent the scaled solution on the fixed mesh through interpolation. This interpolation-resampling strategy was previously proposed in [44] as a means to capture the consequences of space-time translational symmetry on a discrete lattice.

Note 1. If an interpolation method is used, the definition of $f_{n+1}$ in the last step of Algorithm 1 should be modified as follows: $f_{n+1}(x) = L^{\alpha_{n+1}} \frac{||u_n(\cdot, L)||_{\infty}}{||u_n(\cdot, L)||_{\infty}} u_n(x, L)$, where $u_n(x, L)$ is the interpolation of $u_n \left( L^{\beta_n+1} x, L \right)$ onto the fixed mesh. This normalization step eliminates any possible change in the magnitude of $f_{n+1}$ caused by the interpolation.

Finally, to eliminate the need for numerical boundary conditions in our computations of purely initial value problems, typically with compactly supported initial data, a number of additional “ghost” mesh points carrying constant-valued data consistent with the given initial condition are added at the boundary of the computational domain before the spatial scaling is performed.

2.3.1. Positive initial mass. The first initial condition we consider for our numerical experiments is the characteristic function

$$u(x, 0) = \chi_{[-\ell, \ell]}(x) = \begin{cases} 1, & -\ell \leq x \leq \ell, \\ 0, & \text{otherwise}. \end{cases}$$

(17)

Whitham [48] showed that solutions of Burgers equation corresponding to nonnegative initial data with compact support evolve to a special self-similar solution, given by an explicit expression (page 104, equation (4.32)):

$$u(x, t) \sim \sqrt{\frac{2M}{t}} g(z, R), \quad \text{as } t \to \infty.$$  

(18)

Here $M > 0$ is the mass of the initial condition, which is a conserved quantity,

$$M = \int_{-\infty}^{\infty} u(x, t) dx, \quad \text{for } t \geq 0,$$

(19)

$z = x/\sqrt{2Mt}$ is the similarity variable, and $R = M/(2\nu)$ is the Reynolds number, where $\nu$ is the viscosity; the function $g(z, R)$ is

$$g(z, R) = \frac{(e^R - 1)}{2\sqrt{\pi}} \left( e^{-z^2R} - 1 \right) \int_{z\sqrt{R}}^{\infty} e^{-\xi^2} d\xi.$$  

(20)

In view of (18)-(20), we expect the sequences of exponents $\{\alpha_n\}$ and $\{\beta_n\}$ in the nRG calculation to converge to $\alpha = 1/2$ and $\beta = 1/2$. Hence, we start our nRG calculation by fixing $\beta_n = 1/2$. This means that the spatial variable is scaled by $L^{1/2}$ in each iteration. As we mentioned above, this choice keeps the diffusion term $\nu \partial_x^2 u$ in Burgers equation unchanged under the nRG iterations, that is $\nu_n = \nu$ for all $n$. 


With $\beta_n = 1/2$ fixed, we have $\bar{\beta}_n = 1/2$ and $L^{-n\bar{\beta}_n} = L^{-n/2}$. Therefore, looking back at Eq. (8), we recognize that $u_n(L^{-n/2}x, 1) = L^{n\bar{\beta}_n} u(x, L^n)$. Moreover, if the computed exponents $\alpha_n$ converge to $1/2$ (as is the case in our numerical experiments, see below), then $\bar{\alpha}_n \approx 1/2$ for sufficiently large $n$. In this case, with $t = L^n$, we have $L^{n\bar{\alpha}_n} \approx \sqrt{t}$ and so

$$u_n(L^{-n/2}x, 1) \approx L^{n/2} u(x, t) = \sqrt{t} u(x, t) \sim \sqrt{2M} \hat{g}(z, R), \quad t \to \infty.$$  

(21)

Now substituting $z = \frac{\hat{z}}{\sqrt{2M}} = \frac{x}{\sqrt{2M}}$ into $g(z, R)$ to obtain $\hat{g}(\hat{z}, R)$, we rewrite (21) as

$$u_n(L^{-n/2}x, 1) \sim \sqrt{2M} \hat{g}(\hat{z}, R).$$  

(22)

It follows from this relation that plotting the normalized function $\sqrt{2M} \hat{g}(\hat{z}, R)$ (so that its amplitude is one) versus $\hat{z}$ should be equivalent to plotting $u_n$ against $L^{-n/2}x$ at time $t = L^n$.

We are now ready to compare the theoretical similarity profile derived by Whitham [48] (page 106, Figure 4.1) and the one obtained from our nRG procedure. Figure 1 is the comparison of nRG calculations and the theoretical asymptotic solutions. For each of the cases (a), (b), and (c) of Figure 1, the corresponding initial mass is $M = 1, 1$, and $2$ ($\ell = 1/2, 1/2$ and $1$ in Eq. (17)), while the diffusivity is $\nu = 0.01, 0.05$, and $0.01$. The figure shows that the nRG calculations match the theoretical predictions in all cases. In the left frames we plot the final computed profile $f_n$ with the predicted theoretical profile. In the right frames we show the convergence of $A_n$ to its theoretical value $A$, which is given by

$$A = \frac{M}{\int R \phi(x) dx},$$

(23)

where $\phi(x)$ is the computed RG profile. The total number of iterations in each of the cases is $n = 500$. In each iteration, the calculation was carried out in the domain $-8 \leq x \leq 8$, with $\Delta x = 16/5000$, while the time integration was from $t = 1$ to $t = L = 1.2$ with $\Delta t = 0.2/2000$. In Figure 2, we show the calculations of $\alpha_n$ and $\kappa_n$. The figure indicates that $\alpha_n \to \frac{1}{2}$ and that $\kappa_n$ converges to a nonzero value. The convergence of $\kappa_n$ to a nonzero value means that, for the class of initial data considered ($f(x) \geq 0$ with compact support), the nonlinear term $u \partial_x u$ is asymptotically neutral. Thus the long term effects of nonlinearity and diffusion are commensurate.

2.3.2. Zero initial mass. In our next example, we consider the following initial condition for Burgers equation

$$u_0(x) = -\chi[-\ell, 0] + \chi[\ell, 0] = \begin{cases} -1, & -\ell \leq x \leq 0, \\ 1, & 0 < x \leq \ell, \\ 0, & \text{otherwise}. \end{cases}$$

(24)

The mass of this function is zero. For such class of initial conditions (with zero mass), Whitham showed that for small diffusivities, the inviscid theory (i.e., $\nu = 0$) is adequate to describe the way the corresponding solutions of Burgers equation decay to zero during their intermediate asymptotic regime: these solutions resemble $N$-waves for a long time before their final decay. However, for any positive diffusivity, no matter how small, the final decay of such solutions, as $t \to \infty$, is given by

$$u(x, t) \sim \frac{x}{t} \sqrt{\frac{\alpha}{t}} e^{-x^2/(4\nu t)},$$

(25)

for some fixed $\alpha$. The function on the right-hand side of (25) is the dipole solution of the heat equation, which means that diffusion is the dominant effect in the final decay; that is, the nonlinear term is irrelevant.

To study the final decay of solutions for this class of initial data with the nRG algorithm, we fix $\beta = 1/2$ in the spatial scaling. This choice ensures that the coefficient in front of the diffusion term remains unchanged. We set the diffusivity $\nu = 0.01$ and the parameter $L = 1.2$. The total number of iterations is $n = 1500$, and $\ell = 1$ in the initial condition. In Figure 3(a), we display a snapshot of the
Figure 1. Comparison between the nRG calculations and the asymptotic self-similar solutions shown in [48]. The left frames show the computed profiles \( f_n \) in the final iteration \( n = 500 \) and the theoretic predictions for various \( M \) and \( \nu \). The right frames show the convergence of \( A_n \) to its theoretical value \( A \). The initial mass and viscosity used for the comparisons are (a) \( M = 1, \nu = 0.01 \), (b) \( M = 1, \nu = 0.05 \), and (c) \( M = 2, \nu = 0.01 \). All numerical calculations used \( \Delta x = 0.0032 \) and \( \Delta t = 0.0001 \).

nRG solution at the end of the 400th iteration. It is clear that the solution resembles an N-wave at this stage. In Figure 3(b), the final profile of the nRG calculation at \( n = 1500 \) is compared with the dipole solution of the heat equation. The value of \( a \) in (25) is set to \( L^{2n} \). Recall that the scaled spatial variable in the nRG calculation is \( \hat{z} = L^{-n(1/2)} x = x / \sqrt{t} \). Substituting \( a = L^{2n} \) and \( \hat{z} \) into the expression in (25),
Figure 2. $\alpha_n$ (left) and $\kappa_n$ (right) versus $n$ (number of iterations). The initial mass and viscosity are the same as those in Figure 1 (a), (b), and (c). The figure indicates that $\alpha_n \to \frac{1}{2}$ and that $\kappa_n$ converges to a nonzero value, as expected.

and noting that $L^{2n} = t^2$, yields the dipole solution as a function of the similarity variable $\hat{z}$

$$\psi(\hat{z}) = \hat{z} e^{-\hat{z}^2/(4\nu)}.$$  \hspace{1cm} (26)

The circles in Figure 3(b) display this dipole solution, with amplitude normalized to one, plotted against the similarity variable. The result of the nRG calculation at $n = 1500$ (the continuous line in Figure 3(b)) correctly predicts the final decay. The convergence of the scaling parameters $\alpha_n$ and $\kappa_n$ is shown in
Figure 3. Numerical experiment with zero initial mass ($M = 0$). The viscosity is $\nu = 0.01$. (a) Profile $f_n$ resembling an $N$-wave computed by the nRG algorithm in the 400th iteration. (b) Comparison between the asymptotic dipole solution and the nRG calculation at the 1500th iteration.

Figure 4. Renormalized coefficients in the previous calculation (Figure 3). (a) $\alpha_n$ and (b) $\kappa_n$ versus $n$ (number of iterations). Figures 4(a) and (b), respectively. They indicate that $\alpha_n \to 1$ and that $\kappa_n \to 0$, where $\kappa_n$ is the coefficient in front of the nonlinear convection term of the renormalized PDE. The convergence of $\kappa_n$ to zero is evidence that the diffusion term dominates the final decay, regardless the magnitude of the diffusion constant, which remains unscaled throughout the calculation. These figures show the mechanism by which the solution crosses over from the $N$-wave structure to the dipole solution, as different terms become dominant in the equation (see Whitham’s analysis). The sizes of the computational domain, $\Delta x$, and $\Delta t$ are those indicated in Figure 1.

We remark that to prevent numerical artifacts from distorting the symmetry of the solution, we replaced the left hump of the computed profile by the negative mirror image of the right hump at the end of each time evolution, before the scaling transformation was applied.
3. Korteweg-de Vries equation and dispersive shock waves

In this section, we illustrate that the nRG procedure is an efficient method for studying asymptotic behavior of dispersive shock waves (DSWs). DSWs appear when dispersion dominates dissipation for step-like data; they have been observed in plasmas, fluids, superfluids and optics [2]. In a sequence of papers, Ablowitz et al. investigate interactions and asymptotics of DSWs for the Korteweg-de Vries (KdV) equation [1–3]. Their investigation focuses on the KdV equation because it is the leading-order asymptotic equation for weakly dispersive and weakly non-linear systems.

Consider the dimensionless form of KdV equation

\[ u_t + uu_x + \epsilon^2 u_{xxx} = 0, \]

with \( u = u(x,t) \) going rapidly to the far-field conditions

\[ \lim_{x \to -\infty} u(x,t) = 0, \quad \lim_{x \to \infty} u(x,t) = -6c^2. \]

Single stepwise initial data for the above problem evolve to a wedge-shape envelope combining three basic regions: an exponential decay region on the right, a DSW region in the middle, and a region of oscillating tail on the left [2]. All three regions travel to the left with time at a speed \( x = -12c^2t \), while the DSW region is expanding and is of the order \( O(t) \) [2, 3]. The amplitude of the DSWs saturates at \( 6c^2 \).

The KdV equation for the described problem can be posed as an initial-boundary-value problem (IBVP) on a truncated domain \(-\ell < x < \ell\), where the initial and boundary data are prescribed as follows

\[
\begin{align*}
  u_t + uu_x + \epsilon^2 u_{xxx} &= 0, \quad x \in [-\ell, \ell], \quad t > 0, \\
  u(x,0) &= 3(1 - \tanh((x - x_0)/w) - 2), \\
  u(-\ell, t) &= 3(1 - \tanh((-\ell - x_0)/w) - 2), \\
  u(\ell, t) &= 3(1 - \tanh((\ell - x_0)/w) - 2), \\
  u_x(\ell, t) &= 0.
\end{align*}
\]

For example, if \( w = 1 \) and \( \ell - x_0 = 20 \), we have \( u(-\ell, t) \approx 0 \) and \( u(\ell, t) \approx -6 \). The existence and uniqueness of solution of the above IBVP is discussed in [9].

The numerical method we adopt for solving Eq. (29) is a non-oscillatory explicit finite-difference method [47]. The spatial and temporal discretization for the algorithm is

\[
\frac{1}{2} \left( \frac{u_{j-1}^{n+1} - u_{j-1}^n}{\Delta t} + \frac{u_{j+1}^n - u_{j+1}^{n-1}}{\Delta t} \right) = - \left( \frac{u_{j+1}^n + u_j^n + u_{j-1}^n - 3}{\Delta x} \right) u_{j+1}^{n+1} - u_{j-1}^n - \frac{\epsilon^2}{2\Delta x^3} \left( u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n \right).
\]

Applying the von Neumann analysis for the above method yields the stability condition

\[ \frac{\Delta t}{\Delta x} < \frac{2}{\max_{x,t}|u| + 4\epsilon^2/\Delta x}. \]

Adopting the same scaled variables as in Eqs. (3) and (4), the scaled KdV equation is

\[ u_t + L^{1-\alpha-\beta} uu_x + \epsilon^2 L^{1-3\beta} u_{xxx} = 0. \]

Here we have dropped the subscript \( L \) for \( u \) and the tildes for \( t \) and \( x \). Since the amplitude of DSWs saturates at \( 6c^2 \), it is not necessary to scale the amplitude and hence we set \( \alpha = 0 \) for our nRG calculations.

Also, we choose to retain the coefficient of the dispersion term unscaled. This results in fixing \( \beta = 1/3 \) for our nRG calculations, which suggests that the DSW region expands at a rate of order \( O(t^{1/3}) \) for the RG calculations. We choose \( \epsilon^2 = 1 \).

With this set of parameters, Figures 5 and 6 display comparisons between direct numerical simulations and nRG calculations. The initial condition is a single-step tangent profile. The graphs show that the
nRG procedure accurately captures the DSWs in a confined domain and the results are consistent with that in the literature [2, 3]. Moreover, a simple calculation below will illustrate that the nRG procedure is more efficient than direct numerical calculation.

Suppose that the final time for a direct simulation is \( t = L^n \) and the number of solitons in the region of DSWs at the final time is \( N_s \). Assume that the spatial grid-size required to resolve these solitons is \( \Delta x \). For this \( \Delta x \), the required temporal step-size is \( \Delta t = O((\Delta x)^2) \), by the stability condition (31). Therefore the total number of time steps required for the simulation is \( L^n/\Delta t \sim L^n/(\Delta x)^2 \). Now for the nRG procedure, \( t = L^n \) means the number of iterations is \( n \). Since the dispersion coefficient is kept the same, the number of solitons after \( n \) iterations is also \( N_s \) for the nRG procedure. However, because of the spatial scaling, the spatial grid-size required for resolving the solitons is now \( \Delta \tilde{x} = L^{-n\beta} \Delta x \). Hence the temporal step-size for stability requirement is \( \Delta \tilde{t} = O((\Delta \tilde{x})^2) \). The number of time steps for the nRG procedure to the final time is \( n(L-1)/\Delta \tilde{t} \sim n(L-1)L^{2n\beta}/(\Delta x)^2 \). Since \( \beta = 1/3 \), the numerator of the above equation is \( n(L-1)L^{2n/3} \) and this is less than \( L^n \) for large \( n \).

Our numerical experiments also confirm that direct numerical simulation is much more time consuming than the nRG calculation for long time simulations. We further remark that if an implicit algorithm is used, for which \( \Delta t \) could be chosen to be of order \( O(\Delta x) \) [45], then the nRG procedure is even more preferable than direct numerical simulation.

Figure 5. (a) Direct numerical simulation. \( t/\epsilon = 10 \), where \( t = 1.2^8 \). The spatial and temporal step sizes are \( \Delta x = 40/2000 \) and \( \Delta t = 1.2^8/80000 \), respectively. \( x_0 = 20 \) and \( w = 1 \). (b) Numerical renormalization group calculation. \( \beta = 1/3, \alpha = 0, L = 1.2 \), and \( t/\epsilon = 10 \). Eight iterations are performed (\( n = 8 \), i.e. \( t = 1.2^8 \)). For both (a) and (b), the dashed line is the initial condition.

4. A MODIFIED DIFFUSION-ABSORPTION MODEL

In this section, we consider the one-dimensional diffusion-absorption model

\[
 u_t = D(u_t)\partial_x^2 u^{m+1} - \lambda u^p, \tag{33}
\]

where \( \lambda, m \geq 0 \) and \( D(u_t) \) is the (not necessarily constant) diffusion coefficient. The following cases have been considered in the literature:

(1) \( D(u_t) = 1 \) and \( m \geq 1 \). In this case, if \( \lambda > 0 \) and \( p \geq m + 1 \), Eq. (33) is a nonlinear heat equation with absorption [32], whereas if \( 0 < p < 1 \), it is a model for slow diffusion combined with strong absorption [25].
One of our goals here is to compare the long time asymptotics of equations with discontinuous $(\epsilon = 0.1, t = 1.2)$ and $(\epsilon = 0.1, \text{iteration} = 20, L = 1.2)$

Figure 6. (a) Direct numerical simulation. $x_0 = 350$, $w = 1$, $\Delta x = 1/80$, $\Delta t = (1.2)^{20}/32000000 \approx 1.19805 \times 10^{-6}$. (b) Numerical renormalization group calculation. $\beta = 1/3$, $\alpha = 0$, $L = 1.2$, and $\epsilon^2 = 0.1$. Twenty iterations are performed ($n = 20$, i.e. $t = 1.2^{20}$), $x_0 = 150$, $w = 1$, $\Delta x = 1/80$, and $\Delta t = 1 \times 10^{-6}$. For both (a) and (b), the dashed line is the initial condition.

(2) $D(u_t)$ is a step function:

$$D(u_t) = \begin{cases} 1 + \epsilon & \text{for } u_t < 0 \\ 1 & \text{for } u_t > 0. \end{cases}$$

(34)

In this case, if $\lambda = 0$ and $m = 0$, Eq. (33) is the so-called Barenblatt equation [5,20,30], whereas if $m = 1$, it is called the modified porous-medium equation [18].

One of our goals here is to compare the long time asymptotics of equations with discontinuous (step function (34)) $D(u_t)$ with the asymptotics of the same equations but with continuous approximations of $D(u_t)$. Specifically, we shall consider approximations of two types:

(1) $D(u_t)$ is a smooth function interpolating the step between $1 + \epsilon$ and 1, namely,

$$D(u_t) = 1 + \frac{\epsilon}{2 \sigma} \frac{1 - \tanh(u_t/\sigma)}{1 + \tanh(u_t/\sigma)},$$

with $D(u_t) \to 1 + \epsilon$ as $u_t \to -\infty$ and $D(u_t) \to 1$ as $u_t \to \infty$. The parameter $\sigma > 0$ adjusts the width of the transition band of the hyperbolic tangent profile.

(2) $D(u_t)$ is a piecewise linear function interpolating the two constant values $1 + \epsilon$ and 1, namely,

$$D(u_t) = \begin{cases} 1 + \epsilon & \text{for } u_t < -\delta \\ 1 - \frac{\epsilon}{2\delta} (u_t - \delta) & \text{for } -\delta < u_t < \delta \\ 1 & \text{for } u_t > \delta, \end{cases}$$

(36)

where the parameter $\delta$ plays the same role as that of $\sigma$ in Eq. (35).

4.1. Validation of the RG algorithm. We begin by validating the nRG procedure for this class (discontinuous $D(u_t)$) of problems. The self-similar solution of Barenblatt’s equation studied in the literature [20] provides an ideal example. Thus, consider Barenblatt’s equation

$$u_t = D(u_t) u_{xx}, \; x \in \mathbb{R}, \; t > 1,$$

I. C.: $u(x, 1) = u_0(x), \; x \in \mathbb{R},$ (37)
where \( D(u_t) \) is the step function in (34). The long time behavior of Barenblatt’s equation with a rapidly decaying initial condition is given by a similarity solution [20]

\[
\frac{\partial u}{\partial t} = \frac{A}{t^\alpha} \phi \left( \frac{x}{2\sqrt{\kappa t}} \right), \quad \kappa_+ \text{ is the diffusivity for } \frac{\partial u}{\partial t} > 0, \tag{38}
\]

where \( A \) is some pre-factor. For our numerical validation \( \kappa_+ = 1 \) and \( \kappa_- = 1 + \epsilon \). The parameter \( \alpha \) depends on the diffusivity ratio \( \kappa_-/\kappa_+ \). Cole and Wagner [20] obtained this dependency in terms of a perturbation expansion

\[
\alpha(\epsilon) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}\epsilon} \epsilon - 0.06354624 \epsilon^2 + \cdots. \tag{39}
\]

We remark that since the similarity solution of the linear heat equation has the time decay rate \( \alpha = 1/2 \), the \( \epsilon \) and \( \epsilon^2 \) terms in Eq. (39) are sometimes called the anomalous dimension of the decay.

Suppose the time and space variables and the amplitude of \( u \) are scaled as in equations (3) and (4); then the renormalized IVP for the \( n \)th RG iteration takes the form

\[
\partial_t u_n = D \left( L^{-n(1+\bar{\alpha}_n)} \partial_x u_n \right) L^{n(1-2\bar{\beta}_n)} \partial_x^2 u_n, \quad t > 1,
\]

\[
\text{I. C. : } u_n(x, 1) = f_n(x), \tag{40}
\]

where \( \bar{\alpha}_n, \bar{\beta}_n \) and \( f_n(x) \) have been defined in Section 2.1. The numerical solution of these initial value problems, starting with the initial condition

\[
u_0(x, 1) = \begin{cases} \cos x & \text{for } |x| \leq \frac{\pi}{2} \\ 0 & \text{for } |x| > \frac{\pi}{2}, \end{cases} \tag{41}
\]

is performed with the second order Crank-Nicolson method. The diffusivity at the \( (k+1) \)th time step in the \( n \)th RG iteration is approximated by \( D(L^{-n(1+\bar{\alpha}_n)}(3u_n^k - 4u_n^{k-1} + u_n^{k-2})/(2\Delta t)) \). In our nRG calculation, the spatial scaling parameter \( \beta = 1/2 \) is fixed, meaning that the diffusivity values remain unchanged, only their distribution is rescaled (renormalized) upon iteration (see (40)). The time integration in each nRG iteration is from \( t = 1 \) to \( t = L = 1.2 \), and the number of iterations is 100. The domain \(-10 \leq x \leq 10\), the time step \( \Delta t = 0.05 \) and the spatial step \( \Delta x = \frac{20}{160} = 0.125 \) are used for all computations, with several values of the parameter \( \epsilon \). Cubic interpolation is employed for the mesh interpolation approach discussed in Section 2.3.

Figure 7 is the comparison between the nRG computed scaling exponent \( \alpha \) in (38) and its linear and quadratic perturbative values, as predicted by the expansion (39). From this figure we see that for small values of \( \epsilon \), there is very good agreement between the \( \alpha \)-values computed by the nRG algorithm and the perturbation results.

4.2. Non-constant continuous \( D(u_t) \). Suppose now the discontinuous diffusivity (step function (34) with \( \epsilon = 0.5 \)) is approximated by one of two types of continuous functions: (i) the smooth function in (35), with \( \epsilon = 0.5 \) and \( \sigma = 1, 0.5 \) and 0.1 (see Figure 8(a)), or (ii) the piecewise linear function in (36), with \( \epsilon = 0.5 \) and \( \delta = 1, 0.5 \) and 0.1 (see Figure 8(b)). We will demonstrate numerically the behavior of the asymptotic decay exponent \( \alpha \), which crosses over from its \( \epsilon \)-dependent anomalous value to the one of the linear heat equation, when the diffusivity in Barenblatt’s equation is regularized (i.e., approximated by a continuous function). But before we discuss the numerical experiments, we provide a heuristic argument to justify this behavior.

Note that the discontinuous diffusivity remains unchanged under the scaling in (40):

\[
D \left( L^{-n(1+\bar{\alpha}_n)} \partial_x u_n \right) = D(\partial_t u_n) = \begin{cases} 1 + \epsilon & \text{for } \partial_t u_n < 0 \\ 1 & \text{for } \partial_t u_n > 0, \end{cases} \tag{42}
\]

for all \( n \), which means that the value of the diffusivity depends exclusively (and discontinuously) on the sign of \( \partial_t u_n \). The continuous diffusivity functions in (i) and (ii), however, behave differently under the
The time decay exponent $\alpha$ in (38) computed by the nRG algorithm versus the perturbative values predicted by the linear and quadratic approximations, Eq. (39), given in the literature [20].

This simple observation suggests that, since a continuous diffusivity approaches a constant value under repeated renormalizations, the modified Barenblatt equation should behave, asymptotically, like the linear heat equation (with renormalized diffusivity $D = 1 + \epsilon/2$); and, consequently, its time decay parameter $\alpha$ should cross over from its $\epsilon$-dependent anomalous value to $1/2$, i.e., $\alpha_n \to 1/2$.

We now discuss the numerical experiments in which the step-function diffusivity in Barenblatt’s equation is approximated (regularized) by the continuous functions in Figure 8. Figure 9(a) shows the initial profile and the computed asymptotic similarity forms. While the transition band in the three smooth (hyperbolic tangent) diffusivity functions has markedly distinct widths, the computed final asymptotic similarity profiles are visually indistinguishable after 200 nRG iterations. Figure 9(b) shows the time decay parameter $\alpha$ in Eq. (38) for smooth and discontinuous diffusivities during the nRG calculations. As expected, the time decay parameter approaches $1/2$ for the continuous ones.

Figure 10 shows similar calculations for the piecewise linear diffusivity function in Figure 8(b). The results in Figures 9 and 10 are extremely close, despite the two different regularizations (smooth vs. piecewise linear) of the diffusivity discontinuity. Figure 11(a) is the comparison of the computed asymptotic similarity forms for discontinuous (step function) and smooth (hyperbolic tangent) diffusivities. Figure 11(b) displays the diffusivity distributions at the end of the 200 nRG iterations.

4.3. The modified diffusion-absorption model ($\lambda > 0$). We now consider the full diffusion-absorption equation (33). It has been shown that for $D(u_t) \equiv$ constant, $\lambda > 0$, $p > 1 + m$, $m \geq 0$, two different regimes are possible for the decay of solutions to zero: in one, the decay is controlled by diffusion; in the other, it is controlled by absorption [26, 46]. A critical value $p^* = p^*(m, d)$, where $d$ is the spatial dimension of Eq. (33), separates the two regimes. In the range $p > p^*$, the time decay exponent $\alpha$ does
Figure 8. (a) Smooth $D(u_t)$ in Eq. (35) with $\epsilon = 0.5$, $\sigma = 1, 0.5, $ and $0.1$. (b) Piecewise linear $D(u_t)$ in Eq. (36) with $\epsilon = 0.5$ and $\delta = 1, 0.5, $ and $0.1$.

Figure 9. (a) The initial profile and the computed asymptotic similarity forms for Eq. (40). The three examples of smooth $D(u_t)$ shown in Figure 8(a) are used for the calculations. The final asymptotic similarity forms computed after 200 nRG iterations are visually indistinguishable for the three different cases. (b) The time decay exponent $\alpha$ captured by the nRG algorithm for the calculations in (a), compared with that for Barenblatt’s equation, for which $D(u_t)$ is a step function. $\alpha$ seems to converge to 1/2 for the smooth $D(u_t)$ cases after just 20 iterations.

not vary with $p$, depending only on $m$ and $d$, which indicates that for such $p$ the absorption term is an irrelevant perturbation to the diffusion equation, and the asymptotic decay is controlled by diffusion:

$$\alpha = \frac{d}{md + 2}. \quad (44)$$

On the other hand, for $1 + m < p < p^*$, the asymptotic time decay is given by

$$\alpha = \frac{1}{p - 1}, \quad (45)$$

which is the decay imparted by absorption, indicating that for such $p$ the absorption term is a relevant perturbation to the diffusion equation. Moreover, for the marginal (neutral) case, $p = p^*$, the time decay
Figure 10. Same calculation as in Figure 9, but with piecewise linear diffusivity.

Figure 11. (a) A comparison of the computed asymptotic similarity forms for the cases of discontinuous (step function) and smooth ($\sigma = 0.1$) diffusivity. (b) The diffusivity distributions at the end of the 200th nRG iterations. The diffusivity distribution (dashed line) is constant, with value $1 + \frac{\epsilon}{2}$, for the hyperbolic tangent diffusivity function.

is the same for both absorption and diffusion; hence the critical value

$$p^* = m + \frac{2}{d} + 1.$$  \hspace{1cm} (46)

These results are surveyed in, e.g., [43,46].

For the rest of this section, we will consider the cases $m = 0, 1$ with $\lambda = 1$. We will investigate the dependence of the time decay parameter $\alpha$ on the absorption exponent $p$ for distinct choices of non-constant $D(u_t)$, both discontinuous and continuous: (i) $D(u_t)$ is the step function in Eq. (34) with $\epsilon = 0.5, 0$ and $-0.5$; (ii) $D(u_t)$ is the hyperbolic tangent in Eq. (35) with $\sigma = 0.1$ and $\epsilon = 0.5, 0$ and $-0.5$.

Similar to Eq. (40), the scaled Eq. (33) for the $n^{th}$ nRG iteration is

$$\partial_t u_n = D \left( L^{-n(\bar{\alpha}_n+1)} \partial_t u_n \right) L^{-n(\bar{\alpha}_n m + 2\beta_n - 1)} \partial_x u_{m+1}^n - L^{-n(\bar{\alpha}_n(p-1) - 1)} u_{n+1}^p, \ t > 1,$$

I. C.: $u_n(x, 1) = f_n(x)$.

\hspace{1cm} (47)
To solve numerically these initial value problems, we use centered differences for the discretization of the spatial derivative and the explicit Euler’s method for the time evolution. The temporal and spatial step sizes are chosen to satisfy the stability constraint: $D \frac{\Delta t}{(\Delta x)^2} \leq 1/2$. The nonlinear diffusivity $D(u_t)$ is approximated the same way as before.

In all our numerical simulations, the time is integrated from $L = 1$ to $L = 2$ with $\Delta t = 5 \times 10^{-4}$, while the computational domain is $-10 \leq x \leq 10$ with $\Delta x = 0.125$. The initial condition is the one given in Eq. (41). For each simulation, the number of RG iterations is 200.

**Case I:** $m = 0$: We vary $p$ in the interval $2 \leq p \leq 5$ with increment $\Delta p = 0.1$. In order to study the effect of different diffusivities, we keep the magnitude of the diffusivity unscaled throughout the RG iterations by choosing $\beta_n = 1/2$ for all $n$ (see Eq. (47) with $m = 0$).

Figure 12(a) is the plot for the computed time decay parameter $\alpha$ as a function of $p$, when the diffusivity is a step function. The solid line is the theoretically predicted $\alpha$ when $\epsilon = 0$. From Eq. (46), the critical value $p^* = 3$ in this case. Hence for $p < 3$, $\alpha$ obeys Eq. (45) and for $p > 3$, $\alpha = 1/2$ (see Eq. (44)).

The numerically computed values of $\alpha$ for $D(u_t) \equiv 1$ ($\epsilon = 0$) are indicated by circles. The computed values agree with the theoretical prediction. The computed values of $\alpha$ for $\epsilon = 0.5$ and $-0.5$ are indicated by squares and triangles, respectively. The results suggest that $\alpha$ is given by Eq. (45) until a critical $p$ value is reached. For $p$ larger than this critical value, the time decay parameter $\alpha$ is independent of $p$. Moreover, the critical value depends on $\epsilon$.

Figure 12(b) displays the results when the diffusivity is the hyperbolic tangent function. In this case, the time decay parameter $\alpha$ is independent of $\epsilon$ and is given by the theoretical prediction for the case of constant diffusivity ($\epsilon = 0$). This behavior is explained by the heuristic argument presented in Eq. (43).

**Case II:** $m = 1$: Refer to Eq. (47). When $m \neq 0$, the diffusivity values are renormalized in each iteration by a factor that depends on both $\alpha_n$ and $\beta_n$. Thus, to keep the diffusivity unscaled, we now determine the spatial scaling factor $\beta_n$ through the scaling relation $\beta_n = 1 - m\alpha_n$, after $\alpha_n$ is computed at the end of each RG iteration.

The numerical results are shown in Figure 13. Qualitatively, these results are analogous to those in Case I for $m = 0$.

The results of the experiments in Cases I and II suggest the following conjecture:
Figure 13. \( m = 1 \). (a) \( D(u_t) \) is the step function (34) with \( \epsilon = 0.5, 0 \) and \(-0.5\). The solid line is the theoretically predicted \( \alpha \) for \( \epsilon = 0 \). (b) Same as (a), except \( D(u_t) \) is the hyperbolic tangent function (35) with \( \sigma = 0.1 \).

**Conjecture.** For sufficiently localized initial conditions, the solutions of the one-dimensional diffusion-absorption equation (33) with discontinuous diffusivity (34) and \( \lambda > 0 \) decay to zero, as \( t \to \infty \), like \( t^{-\alpha} \), with decay exponent

\[
\alpha = \begin{cases} 
\frac{1}{p-1} & \text{for } 1 + m < p < p^* \\
\alpha(m, \epsilon) & \text{for } p^* < p.
\end{cases}
\]  

(48)

Here \( \alpha(m, \epsilon) \) is the (anomalous) decay exponent of solutions when \( \lambda = 0 \) (pure diffusion), and the critical value \( p^* = 1 + \frac{1}{\alpha(m, \epsilon)} \).

4.4. **Marginal case: vanishing prefactor.** Let us now consider equation (33) with constant diffusivity, \( \lambda > 0 \), and critical (marginal) absorption, i.e., \( p = 3 + m \), see (46). Our intent is to illustrate the remarkable ability of the nRG procedure to uncover and compute correctly logarithmic corrections to asymptotic power-law decay.

For the case \( m = 0, p = 3 \), Bricmont and Kupiainen [14] (among others) proved that the long time behavior of solutions with initial conditions decaying sufficiently fast at infinity is given by

\[
u(x, t) \sim \left( \frac{\lambda \log t}{2 \sqrt{3}} \right)^{-\frac{1}{2}} \phi(x t^{-\frac{1}{2}}),
\]  

(49)

where \( \phi \) is a Gaussian distribution. What happens if we use Algorithm 1 (with \( \beta_n = 1/2 \)) to compute this asymptotic behavior? We may succeed in capturing the power-law decay \( t^{-1/2} \), through the convergence of the computed decay exponents \( \alpha_n \to 1/2 \). But in this case, the sequence of prefactors \( A_n \) should converge to zero. Indeed, looking back at Eq. (10) (with \( B_n = 1 \)), we observe that

\[
L^{n \alpha_n} u(x, L^n) = A_n f_n \left( \frac{x}{L^n \sqrt{2}} \right),
\]  

(50)

from which follows that, as \( t = L^n \to \infty \),

\[
A_n \sim \left( \frac{\lambda \log t}{2 \sqrt{3}} \right)^{-\frac{1}{2}} \to 0.
\]  

(51)

Figure 14(a) exhibits the results of numerical simulations with Algorithm 1. It shows the slow convergence of the computed prefactor \( A_n \) to zero when \( p = 3 \) (the critical value), as suggested by the relation (51). For comparison, the figure also shows the quick convergence of \( A_n \) to a positive value when \( p = 3.05 \) and \( p = 2.95 \) (\( p \) values just above and below the critical one).
Figure 14. (a) Linear diffusion ($m = 0$) with constant diffusivity. $A_n \to 0$ for the marginal case $p = 3$, while $A_n$ converges to a positive value for $p = 3.05$ and $p = 2.95$. (b) Nonlinear diffusion ($m = 1$) with constant diffusivity. $A_n \to 0$ for the marginal case $p = 4$, while $A_n$ converges to a positive value for $p = 4.05$ and $p = 3.95$.

For the case $m = 1$, $p = 4$, Qi and Liu [41] showed that, for initial conditions $u_0$ satisfying
\[
\lim_{|x| \to \infty} \sup |x|^k u_0 < \infty, \quad k > 1,
\]
the long time behavior of solutions is given by
\[
 u(x, t) \sim (t \log t)^{-\frac{1}{3}} \phi \left( \frac{(\log t)^{1/6} x}{t^{1/3}} \right).
\]
So, invoking again Eq. (10), if the scaling exponents $\alpha_n$ and $\beta_n$ computed by Algorithm 1 for this problem converge to the theoretical predictions, namely, $\alpha_n \to 1/3$ and $\beta_n \to 1/3$, then, as $t = L^n \to \infty$, we should have
\[
 A_n \sim (\log t)^{-\frac{1}{3}} \to 0, \quad (54)
\]
\[
 B_n \sim (\log t)^{\frac{1}{6}} \to \infty. \quad (55)
\]

In Figure 15, we show the convergence of the nRG computed scaling exponents to their theoretical values in Eq. (53). In Figure 14(b), we show that the computed prefactor $A_n$ continues to approach zero after 50,000 nRG iterations, as suggested by the relation (54). For comparison, this figure also shows the fast convergence of $A_n$ to nonzero values, for $p = 4.05$ and $p = 3.95$, which differ slightly from the critical value, $p = 4$. Similarly, the computed prefactor $B_n$ increases for $p = 4$ and quickly settles into a nonzero constant for $p = 4.05$ and $p = 3.95$ in Figure 16(a). Moreover, according to the solution (53), we expect that, in the log-log scale, $A_n$ decays at the rate of $1/3$ and $B_n$ grows at the rate of $1/6$, as shown in Figure 17. More detail about the decay (growth) rates will be provided in the next section.

Note 2. We remark that certain aspects of the asymptotic behavior under scrutiny are uncovered by the nRG algorithm, Algorithm 1, when the numerical convergence of the monitored quantities, such as the scaling exponents $\alpha_n$ or $\beta_n$ in Figure 15, are observed through the iterations. However, we have to be cautious, in the sense that a conclusion must not be drawn until we look at the convergence of all other monitored quantities: prefactors ($A_n$ and $B_n$), profiles $f_n$, and any rescaled PDE coefficients (e.g., $D_n$, $\lambda_n$ in Eq. (57)). Let us elaborate this observation using the previous study of the marginal (critical) absorption as an example.
Consider the nonlinear diffusion-absorption equation
\[ u_t = D \partial_x^2 u^{m+1} - \lambda u^p, \quad (56) \]
where \( D \) and \( \lambda \) are constant. If we assume the scaling transformation (3), after \( n \) scaling transformations, the equation becomes
\[ \partial_t u_n = D_n (\partial_x^2 u_n)^{m+1} - \lambda_n u_n^p, \quad t > 1, \quad (57) \]
where
\[ D_n = DL^{-n(\alpha_n m + 2\beta_n - 1)} \quad \text{and} \quad \lambda_n = \lambda L^{-n(\alpha_n (p-1) - 1)}. \quad (58) \]
For \( m = 1, p = 4 \), applying the nRG algorithm we observe that \( \alpha_n \to 1/3 \) and \( \beta_n \to 1/3 \), as Figure 15 indicates. However, we should not conclude that the asymptotic behavior is of the form given in (10), because the prefactors \( A_n \) and \( B_n \) do not converge to some nonzero values, as Figures 14(b) and 16(a) indicate. The behavior of \( A_n \) and \( B_n \) suggest that there may be hidden decay or growth factors in the asymptotics. Furthermore, we not only know that there are hidden decay or growth rates from the log-log
plots of $A_n$ and $B_n$, but we also know that there is a decay rate of $(\log t)^{-1/3}$ in the amplitude and a growth rate of $(\log t)^{1/6}$ in the spatial scaling by looking at the log-log plots of $A_n$ and $B_n$ (see Figure 17). This demonstrates the nRG algorithm is capable of uncovering hidden information and guiding the analysis in the right direction.

We note that, in this nRG calculation, the scaling exponents $\beta_n$ are determined from the scaling relation $\bar{\alpha}_n m + 2\bar{\beta}_n - 1 = 0$, so that $D_n = D$ for all $n$. This choice (to fix the diffusivity $D$) is guided by our desire to study how solutions depend on the absorption power $p$. In each iteration, after we compute $\alpha_n$, we obtain $\beta_n$ through this scaling relation. After 50,000 nRG iterations, we observed that $\alpha_n \to 1/3$ and $\beta \to 1/3$.

Finally, we comment that the numerical experiments in Figures 14 and 16 use the compactly supported initial condition (41) with $D \equiv 1$ and the normalized absorption coefficient $\lambda = 1$. The parameters for the nRG iterations are $L = 2$, $\Delta x = 0.1$, and $\Delta t = 10^{-4}$. Similar to the previous experiments, the spatial derivative is approximated by second-order finite differences and Euler’s method is used for the time evolution.

The peculiar phenomenon observed in the above study of the marginal case motivates us to modify our Algorithm 1 in the next section, in order to capture the hidden logarithmic time decay exponent illustrated in this section.

5. Cubic autocatalytic chemical reaction system

The nRG procedure described in Algorithm 1 assumes that the asymptotic solutions decay or expand at a rate obeying the power law. However, there are differential equations (or systems of differential equations) whose solutions decay at a rate other than the power law, such as the logarithmic decay discussed in the previous Section. For these solutions, the aforementioned Algorithm 1 is not sufficient to capture the correct decay at the asymptotic region. Nevertheless, the procedure could provide sufficient information that allows us to modify the current algorithm to capture the similarity solutions of those equations.

To illustrate the modification, we consider the Cauchy problem of the chemical reaction system

$$u_t = u_{xx} - u^p v^q,$$

$$v_t = d v_{xx} + u^p v^q,$$

\begin{align}
\text{(59)}
\end{align}
where \( p + q = 3, 1 \leq p, q \leq 2 \), and \( d > 0 \). This system arises as a model for cubic autocatalytic chemical reactions of the type

\[
pK_1 + qK_2 \rightarrow 3K_2
\]

with isothermal reaction rate proportional to \( u^p v^q \), where \( u \) is the concentration of reactant \( K_1 \) and \( v \) is the concentration of auto-catalyst \( K_2 \) [37]. The system is subject to the initial data \( u(x, 0) = a_1(x) \) and \( v(x, 0) = a_2(x) \), where \( a_1, a_2 \geq 0 \) and \( a_1, a_2 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). The above system has been used to model thermal-diffusive combustion problems [13] and mathematical biology [23].

Li and Qi extend the above result by considering the values \( 1 < p, q < 2 \) and \( p + q = 3 \). The nontrivial initial data \( a_i \geq 0 \), for \( i = 1, 2 \) are the same as before, whereas the total mass \( A = \int_{\mathbb{R}} (u(x, 0) + v(x, 0)) dx \), which is a conserved quantity. Li and Qi show that

\[
\sqrt{t} (\log t)^{(p-1)/2} u(\sqrt{t} x, t) \rightarrow B \phi_1(x),
\]

\[
\sqrt{t} v(\sqrt{t} x, t) \rightarrow A \phi_d(x),
\]
as \( t \rightarrow \infty \), where

\[
\phi_d(x) = \frac{1}{\sqrt{4\pi d}} e^{-x^2/4d}
\]

and

\[
B = \left( 4\pi d^{q/2} (p + q/d)^{1/2} \right)^{1/(p-1)},
\]

and \( \phi_1 \) is \( d = 1 \) in Eq. (62). The peculiar phenomena of the similarity solution (61) is that the \( u \)-component contains two decays, the regular power-law decay and a logarithmic decay.

We illustrate below that the nRG algorithm described in Section 2.2 is not sufficient to capture the second decay. However, the procedure will provide a clue for the existence of the second decay, and allows us to design a nRG procedure to capture the similarity solutions in Eq. (61). To this end, we start with the regular nRG procedure stated in Section 2.2, i.e. the scaling for \( t \) and \( x \) is the same as that in Eq. (3), which results in \( u \) and \( v \) being scaled by

\[
u_L(\tilde{x}, \tilde{t}) = L^{\alpha_1} u(x, t) = L^{\alpha_1} u(L^{\beta_1} \tilde{x}, L\tilde{t}),
\]

\[
v_L(\tilde{x}, \tilde{t}) = L^{\alpha_2} v(x, t) = L^{\alpha_2} v(L^{\beta_2} \tilde{x}, L\tilde{t}).
\]

Thus the scalings result in a system of PDEs (dropping the subscript \( L \) and \( \tilde{\cdots} \))

\[
\begin{aligned}
  u_t &= -L^{-2\beta_1 + 1} u_{xx} - L^{1-\rho a_1-\rho a_2+\alpha_1} u^p v^q, \\
  v_t &= -L^{-2\beta_2 + 1} v_{xx} + L^{1-\rho a_1-\rho a_2+\alpha_2} u^p v^q.
\end{aligned}
\]

Similar to the Burgers equation, we choose to keep the diffusion coefficients invariant. Thus \( \beta_1 = \beta_2 = 1/2 \) at all time, whereas \( \alpha_1 \) and \( \alpha_2 \) are computed by step (2) in the nRG algorithm described in Section 2.2, respectively. With this choice of \( \beta_1 \) and \( \beta_2 \), the scaled PDE at the \( n^{th} \) iteration is

\[
\begin{aligned}
  \partial_n u_n &= \partial_x^2 u_n - L^n \left( L^{-n\alpha_1,n} \right)^{p-1} \left( L^{-n\alpha_2,n} \right)^q u_n^p v_n^q, \\
  \partial_n v_n &= d \partial_x^2 v_n + L^n \left( L^{-n\alpha_1,n} \right)^{p} \left( L^{-n\alpha_2,n} \right)^{q-1} u_n^p v_n^q,
\end{aligned}
\]

where \( \alpha_{1,n} \) and \( \alpha_{2,n} \) are defined as \( \alpha_n \) in Section 2.1. At this stage, we assume that the power-law scaling, based on the hypothesis, is

\[
\begin{aligned}
  u(x, L^n) &\sim \frac{A_u}{L^{n/2}} \phi_u(\frac{x}{L^{n/2}}), \\
  v(x, L^n) &\sim \frac{A_v}{L^{n/2}} \phi_v(\frac{x}{L^{n/2}}),
\end{aligned}
\]

where \( A_u \) and \( A_v \) are non-zero constant.
The nRG iteration, based on the power-law decay assumption, in principle will show that, as $n \to \infty$, $A_{u,n} = L_n^{(\alpha_1,n - \bar{\alpha}_1,n)} \sim A_u$, and $A_{v,n} = L_n^{(\alpha_2,n - \bar{\alpha}_2,n)} \sim A_v$ (see Eq. (11)). Unfortunately (or fortunately), this is not the case. The numerical experiment, in fact, shows that $A_{u,n} \to 0$, while $A_{v,n} \neq 0$ and converges to some constant proportional to the total mass $A_{\text{total}}$, as $n \to \infty$. Based on this result, we conjecture that the $v$-component follows the power-law decay, similar to the Burgers equation, while the $u$-component has a hidden decay that is not captured by solely assuming the power-law decay.

5.1. A numerical experiment. We conduct a nRG experiment using the power-law scaling, Eq. (67), for the above chemical reaction problem with the parameters, $p = q = 1.5$, $d = 0.75$, and $L = 1.2$. The initial data are

$$u(x, 0) = v(x, 0) = \chi_{[-\ell, \ell]}(x) = \begin{cases} 1, & -\ell \leq x \leq \ell, \\ 0, & \text{else}. \end{cases}$$

(68)

We choose $\ell = 0.5$ and the computational domain to be $[-10, 10]$. For these initial data, the total conserved mass is $A = 2$. Figure 18(a) is a plot for $\alpha_1,n$ and $\alpha_2,n$ versus $n$. From the figure, we expect that $\alpha_1,n$ and $\alpha_2,n$ both converge to $1/2$ as $n \to \infty$, although the figure suggests that $\alpha_1,n$ may converge much slower than $\alpha_2,n$. Since $\beta_n = 1/2$ for all $n$, from Eq. (11), the convergences of $\alpha_1,n$ and $\alpha_2,n$, leads to Eq. (67). Moreover, Figure 18(b) are the computed $A_{u,n}$ and $A_{v,n}$. As expected, $A_{u,n}$ (the dashed-line) approaches $0$ as $n \to \infty$, while $A_{v,n}$ approaches a constant $A_v$. Note that from Eqs. (61) and (67), with $t = L^n$, we have

$$A_{v,n} \phi_v \to A_v \phi_v = A \phi_d.$$  

(69)

Since $\phi_v = \sqrt{4\pi d} \phi_d$, $\sqrt{4\pi d} A_v = A$, or $A_v = \frac{1}{\sqrt{4\pi d}} A$. For $d = 0.75$, $A = 2$, $A_v \approx 0.6515$; i.e., $A_{v,n} \to A_v \approx 0.6515$, and this is exactly what we observe in Figure 18(b).

Figure 19 shows the comparison between the computed Gaussian similarity profile and the predicted theoretical profile in [37] at $n = 3000$, after adjusting the amplitudes. Both components correctly match the prediction, even though the log-logarithmic decay in $u$ is not captured by the nRG algorithm.

Now let us turn our attention to $A_{u,n}$. The fact that $A_{u,n} \to 0$ as $n \to \infty$ indicates that there was a “hidden” decay factor that was not captured by the current nRG procedure. Plotting $A_{u,n}$ vs. $n$ in log-log scale, Figure 20 shows that $\log A_{u,n} = (-2)(\log n) + \log C$, as $n \to \infty$. If we suppose that the hidden decay factor is related to $\log t$, then we could choose $C$ to be $C = A(\log L)^{-2}$. This results in $A_{u,n} = A(\log L^n)^{-2}$, and thus Eq. (67) becomes

$$u(x, L^n) \sim \frac{A}{L^{n/2}(\log L^n)^2} \phi_u\left(\frac{x}{L^{n/2}}\right),$$

$$v(x, L^n) \sim \frac{A_v}{L^{n/2}} \phi_v\left(\frac{x}{L^{n/2}}\right).$$

(70)

Eq. (70) is evidently the (asymptotic) similarity solution of the system of chemical-reaction equations for $t = L^n$ and $p = q = 3/2$ (see Eq. (61)).

6. Modified RG algorithm for logarithmic decay

The above experiment suggests that the component $v$ has the decay factor $\sqrt{t}$, but the component $u$ may have more than one decay factor. Without the asymptotic formula (61), in principle, we do not know what the decay factors are. However, if we hypothesize that one of them is also $\sqrt{t}$ based on Figures 18 and 20, and suppose that the other is related to $\log t$ with some unknown power $\gamma$, then the solution in the asymptotic region gives us an idea of how to compute the power $\gamma$ at the end of each iteration. Taking the hint from Eq. (61), at times $t$ and $L_t$, the ratio of the solutions is

$$\frac{||u(x, t)||_{\infty}}{||u(x, L_t)||_{\infty}} = L^{1/2}t^{1/2}(\log L t)\gamma = L^{1/2}\left(\frac{(\log L t)}{(\log t)}\right)^\gamma$$

(71)
Figure 18. Computed scaling factors by the nRG procedure stated in Section 2.2 for the chemical reaction system. (a) $\alpha_{1,n}$ and $\alpha_{2,n}$ (b) $A_{u,n}$ and $A_{v,n}$.

Figure 19. Comparison between the computed Gaussian similarity profile by Algorithm 1 and the predicted theoretical profile in [37] at $n = 3000$, after adjusting the amplitudes. (a) $u$-component, (b) $v$-component.

To modify the nRG procedure for this case, we observe that at the end of the $(n-1)^{th}$ iteration $t = L^n$,

$$L^{1/2} \left( \frac{\log L^n}{\log L^{n-1}} \right)^{\gamma_n} = L^{1/2} \left( \frac{n}{n-1} \right)^{\gamma_n} = \frac{||u_{n-1}(\cdot, 1)||_{\infty}}{||u_{n-1}(\cdot, L)||_{\infty}}, \quad n > 1,$$

following Eq. (71). Note that for the case $n = 1$, $\gamma_1$ is computed by the power-law scaling,

$$L^{\gamma_1} = \frac{||u_0(\cdot, 1)||_{\infty}}{||u_0(\cdot, L)||_{\infty}}.$$

Eqs. (72) and (73) suggest that for the $u$-component, the initial condition for the next iteration is set by

$$u_n(x, 1) = L^{1/2} \left( \frac{n}{n-1} \right)^{\gamma_n} u_{n-1}(L^{1/2} x, L), \quad \text{for } n > 1,$$

and

$$u_1(x, 1) = L^{\gamma_1} u(L^{1/2} x, L), \quad \text{for } n = 1.$$
Here we have chosen $\beta_1 = \beta_2 = 1/2$ in order to keep the diffusion coefficients unchanged. Note that from Eq. (74), at the end of the $n^{th}$ iteration ($n > 1$), the iterative solutions $u_n$ and $v_n$ are related to the solutions of the PDE’s by

$$
\begin{align*}
  u_n(x,t) &= L^{\gamma_1+(n-1)/2} \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{\gamma_k} u(L^{n/2}x, L^{n}t), \\
  v_n(x,t) &= L^{\bar{\alpha}_2,n}v(L^{n/2}x, L^{n}t),
\end{align*}
$$

(76)

where $\bar{\alpha}_{2,n} = (\alpha_{2,1} + \cdots + \alpha_{2,n})/n$. Eq. (76) implies that

$$
\begin{align*}
  u(x,t) &= L^{-\gamma_1-(n-1)/2} \prod_{k=2}^{n} \left( \frac{k-1}{k} \right)^{\gamma_k} u_n(L^{-n/2}x, L^{-n}t), \\
  v(x,t) &= L^{-n\bar{\alpha}_2,n}v_n(L^{-n/2}x, L^{-n}t).
\end{align*}
$$

(77)

Hence for the $n^{th}$ iteration ($n \geq 1$), the scaled system of PDEs for $u_n$ and $v_n$ is

$$
\begin{align*}
  \partial_t u_n &= \partial_{x}^2 u_n - L^n L^{-(p+1)\gamma_1} (L^{-(n-1)/2})^{(p-1)} \left( \prod_{k=2}^{n} \left( \frac{k-1}{k} \right)^{\gamma_k} \right)^{p-1} (L^{-n\bar{\alpha}_2,n})^{p} u_n^{p} v_n^{q}, \\
  \partial_t v_n &= d \partial_{x}^2 v_n + L^n L^{-p\gamma_1} (L^{-(n-1)/2})^{p} \left( \prod_{k=2}^{n} \left( \frac{k-1}{k} \right)^{\gamma_k} \right)^{p} (L^{-n\bar{\alpha}_2,n})^{q-1} u_n^{p} v_n^{q},
\end{align*}
$$

(78)

for $\beta_1 = \beta_2 = 1/2$. For $n = 0$, the unscaled equation (59) is solved.

We define a variable $A_{u,n}$, a prefactor similar to (11), for the $u$-component, so that we can monitor $A_{u,n}$ for convergence. From the first equation in Eq. (70) and the first equation in Eq. (76), we have

$$
\begin{align*}
  u(L^{n/2}x, L^{n}) &\sim L^{-n/2}(\log L)^{-\gamma} A\phi(x), \\
  u_n(x,1) &= L^{\gamma_1+(n-1)/2} \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{\gamma_k} u(L^{n/2}x, L^{n}).
\end{align*}
$$

(79)

If we let $A_* = A(\log L)^{-\gamma}$ and assume that $\gamma_n \to \gamma$ as $n \to \infty$, Eq. (79) implies

$$
\begin{align*}
  u_n(x,1) &\sim L^{\gamma_1+(n-1)/2} \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{\gamma_k} L^{-n/2} A_*^{x} \phi(x) \\
  &\sim L^{\gamma_1-1/2} \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{\gamma_k-\gamma_n} A_*^{x} \phi(x),
\end{align*}
$$

(80)
Algorithm 2: The modified nRG procedure for the chemical reaction system

for $n = 0, 1, 2, \ldots$, until convergence do

1. Start with the IVP (59) for $n = 0$. Evolve $u_n$ and $v_n$ from $t = 1$ to $t = L$, using the IVP (78) for $n \geq 1$.
2. Compute $\gamma_n$ for the $u$-component by
   
   $L^{\gamma_n} = \frac{||u_0(\cdot, 1)||_{\infty}}{||u_0(\cdot, L)||_{\infty}},$
   $L^{1/2} \left( \frac{n}{n-1} \right)^{\gamma_n} = \frac{||u_{n-1}(\cdot, 1)||_{\infty}}{||u_{n-1}(\cdot, L)||_{\infty}}, \quad n \geq 2.$

   Compute $\alpha_{2,n}$ for the $v$-component by
   
   $L^{\alpha_{2,n}} = \frac{||v_{n-1}(\cdot, 1)||_{\infty}}{||v_{n-1}(\cdot, L)||_{\infty}}.$

3. Compute $A_{u,n} = L^{1/2 - \gamma_n} \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{\gamma_n - \gamma_k}, \quad n > 1; \quad A_{v,n} = L^{\alpha_{2,n} - \alpha_{2,n}}$, where $\bar{\alpha}_{2,n} = (\alpha_{2,1} + \cdots + \alpha_{2,n})/n$.

4. Set initial data for the next iteration by
   
   $f_{u,n+1} = L^{1/2} \left( \frac{n}{n-1} \right)^{\gamma_n} u_n(L^{1/2} x, L)$
   $f_{v,n+1}(x) = L^{\alpha_{2,n} v_n(L^{1/2} x, L)}.$

end for

The above relations hold, since $\gamma_n \rightarrow \gamma$ and $n^{-\gamma_n} = \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{-\gamma_n}$ for $n \rightarrow \infty$. Eq. (80) is equivalent to

$L^{1/2 - \gamma_n} \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{\gamma_n - \gamma_k} u_n(x, 1) \sim A_e \phi(x).$ \hspace{1cm} (81)

If we define

$A_{u,n} = L^{1/2 - \gamma_n} \prod_{k=2}^{n} \left( \frac{k}{k-1} \right)^{\gamma_n - \gamma_k}, \quad n > 1,$ \hspace{1cm} (82)

we expect that $A_{u,n} \rightarrow A_e$ for $n$ large enough, provided $u_n(x, 1) \rightarrow \phi(x)$. Since $\phi = \sqrt{4\pi} \phi_1$, where $\phi_1$ is the Gaussian function in Eq. (62) with $d = 1$, this implies that $A\sqrt{4\pi} = B$, where $B$ is the theoretical prediction in Eq. (63). Therefore

$A_{u,n} \rightarrow A_e = A(\log L)^{-\gamma} = \frac{B(\log L)^{-\gamma}}{\sqrt{4\pi}}.$ \hspace{1cm} (83)

For $A_{v,n}$ we expect $A_{v,n} \rightarrow A_v = \frac{A}{\sqrt{4\pi}}$, where $A$ is the conserved total mass, the same as before. We summarize the modified nRG procedure for the chemical reaction problem with the choice of parameters $\beta_1 = \beta_2 = 1/2$ in Algorithm 0.

7. A Numerical Experiment for the Logarithmic Decay

To illustrate that the modified RG algorithm accurately captures the logarithmic decay, we apply Algorithm 0 to the reaction-diffusion equations (59) with $p = q = 3/2$. For this set of parameters, the large-time asymptotical behavior of the solutions follows Eqs. (61) and (63). Thanks to the choice of $\beta_1 = \beta_2 = 1/2$, the diffusivities in Eq. (78) are kept to be 1 and $d$, respectively. Here we choose $d = 3/4$. The system of PDEs (78) is discretized by an explicit second-order method (forward Euler for the time derivative and the second-order center difference for the spatial second derivative).
The parameters used in our nRG algorithm are $L = 1.25$, $x \in [-10, 10]$, $\Delta x = 0.04$, and $\Delta t = 0.00025$. The number of iterations for nRG is 3000. The theoretical prediction for the critical exponents is $\gamma = 2$ (power of the logarithmic decay for $u$) and $\alpha = 0.5$ (power of the power law decay for $v$). Figure 21 shows that Algorithm 0 accurately captures these two exponents. In the mean time, the theoretical prediction for the pre-factor $A_\ast$ is $A_\ast \approx 1016.89$, computed by Eq. (83), and the pre-factor $A_v$ is $A_v \approx 0.6515$, the same as our previous calculation, and we observe from Figure 22 that both $A_{u,n}$ and $A_{v,n}$ numerically converge to their respective theoretical values.

Finally, in the Section 5.1, our calculation suggests that the original nRG algorithm captures the power law exponents and produces the final similarity profiles that match the theoretical prediction in [37] without taking into account the logarithmic decay. In this experiment, we use the hint from the previous calculation to assume the exponent of the power law decay for the $u$ component. We modified the RG algorithm to include the logarithmic decay. The modified RG algorithm captures the critical exponents and render the numerically convergent prefactors for both components.

It remains to show whether the similarity profiles produced by the modified RG algorithm match the theoretical prediction. To this end, Figure 23 shows that the modified RG algorithm produces the similarity profiles that match the theoretical prediction exactly after we adjust the amplitudes by multiplying the factor $1/\sqrt{4\pi d}$, where $d = 1$ for $u$ and $d = 3/4$ for $v$, respectively.

Note 3. There is no analysis for the stopping criterion for the proposed nRG algorithms. For all our numerical experiments, we stop our iterations when a criterion normally used for linear convergence is satisfied. For example, for prefactors (or decay parameters) we stop the iteration when

$$\left| \frac{A_n - A_{n-1}}{A_{n-1}} \right| < \epsilon$$

for some $\epsilon \sim 10^{-7}$.

![Figure 21](image)

**Figure 21.** Comparison of the theoretical prediction and the nRG computation for the exponents: (a) the sequence of logarithmic decay exponent $\gamma_n \to \gamma = 2$ for $u$, and (b) the sequence of the power law decay exponent $\alpha_n \to \alpha = 0.5$ for $v$.

### 8. Concluding Remarks

We have presented and systematically examined a numerical procedure, based on the RG theory for PDEs, that renders the detailed and efficient computation of asymptotically self-similar dynamics in solutions of PDEs. The effectiveness and robustness of the nRG algorithms were illustrated through several
Figure 22. Comparison of the theoretical prediction and the nRG computation for the pre factors: (a) $A_{u,n} \rightarrow A_u \approx 1016.89$, and (b) $A_{v,n} \rightarrow A_v \approx 0.6515$.

Figure 23. Comparison between the computed Gaussian similarity profile by Algorithm 0 and the predicted theoretical profile in [37] at $n = 3000$, after adjusting the amplitudes. (a) $u$-component, (b) $v$-component.

examples of quasilinear and nonlinear PDEs combining diffusive, reactive and nonlinear propagation effects. It is worth noting that the modified RG algorithm presented in Sections 6 and 7 for the nonlinear system of cubic autocatalytic chemical reaction equations nicely responds to the remark made by Li and Qi [37]:

“The appearance of $\log t$ indicates the analysis is more involved and subtle. In particular, it is well known in the scientific computation field that a scaling of $\log t$ is hardly detectable in computation.”

by detecting the extra decay and capturing the power of logarithmic decay. We refer readers to [34] for some results of the calculations of multidimensional problems by using the similar numerical scaling strategy described in this paper. A proper modification of the described RG algorithm can be used to compute traveling waves and is currently under our investigation. We are also investigating the applicability of an adapted version of the RG algorithm to blow-up problems.
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