QUANTIFYING EXPONENTIAL GROWTH:
THE CASE OF THE JACTUS

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Abstract
This article presents the results of a small-scale teaching experiment with three middle school students who explored exponential growth by reasoning with the co-varying quantities height and time. Three major conceptual shifts occurred during the course of the teaching experiment: a) From repeated multiplication to a coordination of growth in height and time values; b) From coordinating height and time to coordinating constant ratios; and c) Generalizing to non-natural exponents. The details of each of the three shifts is explored, followed by a discussion of the implications of addressing exponential growth from a covariation of quantities perspective.

The Landscape of Exponential Functions in Common School Treatments

Students’ mathematical learning is the reason our profession exists. Everything we do as mathematics educators is, directly or indirectly, to improve the learning attained by anyone who studies mathematics. (Thompson, 2008, p. 45)

Exponential functions are an important concept both in school algebra and in higher mathematics. Not only do they play a critical role in college mathematics courses such as calculus, differential equations, and complex analysis (Weber, 2002), they also represent an important transition from middle school mathematics to the more complex ideas students encounter in high school mathematics. A focus on the conceptual underpinnings of exponential growth has increased in recent years; for instance, the Common Core State Standards (CCSS) highlight the need to understand exponential functions in terms of one quantity changing at a constant percent rate per unit interval relative to another. Moreover, these ideas are also being pushed down into middle school mathematics courses, both in terms of national standards such as the CCSS as well as in middle school curricula (e.g., Lappan et al., 2006).

The study we report on in this paper is situated in a larger project exploring middle-school students’ understanding when reasoning with co-varying quantities to support their development of function understanding. Our prior studies focused on students’ understanding of linear function and quadratic function as they reasoned with quantitative relationships such as gear ratios, constant speed, and growing rectangles (Ellis, 2007, 2011a, 2011b). A natural extension of this work is to explore how students come to understand exponential growth through reasoning in a similar context, namely, by exploring two co-varying quantities such that one changes exponentially as the other changes linearly. Our approach differed in significant ways from the typical textbook approaches to exponential growth, which we detail below.
The Repeated Multiplication Approach
A common textbook treatment for introducing the notion of exponential growth is the repeated multiplication approach. For instance, in the middle school curriculum *Connected Mathematics Project* (2006), students place coins called rubas on a chessboard in a doubling pattern, and then use tables, graphs, and equations to examine the exponential relationship between the number of squares and the number of rubas. These types of tasks require students to perform repeated multiplication to solve a problem and then connect that process to exponential notation (Castillo-Garsow, submitted). A number of researchers have advocated for this approach, suggesting that we define exponentiation as repeated multiplication with natural numbers, and then help students generalize beyond the natural numbers (e.g., Goldin & Herscovics, 1991; Weber, 2002). However, this approach has its limitations. As Davis (2009) noted, generalizing to non-natural exponents may pose difficulties for students; for instance, an expression such as $2^{1/2}$ can be difficult to understand from a repeated-multiplication perspective.

Difficulties in Understanding Exponential Growth
The literature on students’ and teachers’ understanding of exponential growth is scant, but the research that does exist supports Davis’ concerns about the difficulties in generalizing one’s understanding of exponentiation as repeated multiplication. For instance, Weber (2002) found that college students struggled to understand or explain the rules of exponentiation and could not connect them to rules for logarithms. Weber described students’ difficulties in explaining what a function such as $f(x) = a^x$ meant, as well as in explaining why a function such as $f(x) = \left(\frac{1}{2}\right)^x$ was a decreasing function. Pre-service teachers have not fared much better; researchers have identified their struggles not only in understanding exponential functions, but also in recognizing growth as exponential in nature (Davis, 2009; Presmeg & Nenduradu, 2005). Although pre-service teachers appear to have a strong understanding of exponentiation as repeated multiplication, they experience difficulty in connecting this understanding to the closed-form equation and in appropriately generalizing rules such as the multiplication and power properties of exponents (Davis, 2009). In general, teachers appear to be able to make some use of graphical, algebraic, and tabular representations, but cannot then leverage their algebraic facility to support their ability to solve exponential problems or to translate from table situations to either recursive representations or correspondence rules (Davis, 2009; Presmeg & Nenduradu, 2005).

Research on middle school and high school students reveals difficulties as well; students struggle to transition from linear representations to exponential representations, or to identify what makes data exponential (Alagic & Palenz, 2006). In general, exponential growth appears to be challenging to represent for both students and teachers, and it is difficult for teachers to both anticipate where students might struggle in learning about exponential properties and develop ideas for appropriate contexts that involve exponential growth (Davis, 2009; Weber, 2002). These documented challenges suggest a need for better understanding of how to foster students’ learning about exponential growth, and for identifying more effective modes of instruction on exponential functions.

Alternate Approaches to Exponential Growth
Repeated multiplication is not the only way to think about exponential growth; one can also approach exponentiation in other ways, for instance, as the relationship between a population of individuals and their collective growth contributions (Castillo-Garsow, submitted), as prod-
ucts of factors (Weber, 2002), or as a multiplicative rate constructed from multiplicative units (Confrey & Smith, 1994; 1995). Weber (2002) offered a theoretical analysis of exponential growth relying on Dubinsky’s (1991) APOS (Action, Process, Object, Schema) theory. Although this approach begins with an action understanding of exponentiation as repeated multiplication, Weber offers a vision of students then transitioning to a process understanding by interiorizing the repeated multiplication action; students would then view exponentiation as a function and be able to reason about its properties. Once students can consider exponential expressions as a result of a process, terms such as $2^3$ can be viewed as the product of 3 factors of 2, and ultimately students should then be able to generalize their understanding to view $a^b$ as $b$ factors of $a$. Weber’s analysis offers a vision for moving beyond the repeated multiplication view, but it remains an open question how students might actually undergo these processes.

Confrey and Smith (1994, 1995) introduced an operational basis for multiplication and division called splitting, which is a multiplicative operation that is not repeated addition. A splitting structure is a multiplicative structure in which multiplication and division are inverse operations, such as repeated doubling and repeated halving. Within this model, students also treat the product of a splitting action as the basis for its reapplication; thus, a split can be viewed as a multiplicative unit. Confrey and Smith (1994) assert that “Building concepts of multiplicative rates constructed from multiplicative units should play a central role as students work on understanding how multiplicative worlds generate constant doubling times and constant half-lives.” (p. 55)

Splitting as an operation can form the basis of a rate of change approach to exponential functions, which we will discuss in further detail below. In Confrey and Smith’s (1994) work, they found a number of different rate-of-change approaches adopted by students making sense of exponential situations, including multiplicative rates of change. Students constructing multiplicative rates of change would interpret a table with, for instance, a growth factor of 9 to be increasing by “a constant rate of nine.” Confrey and Smith suggest that this is an important conception of rate of change, which is found by calculating the ratios between succeeding $y$-values for constant unit-change in the $x$-values. We highlight this conception as an important foundational idea for a rate-of-change approach to exponentiation.

The Rate of Change Approach to Exponential Functions: Covariation and Continuous Variation

Traditional approaches to function rely on a correspondence view (Smith, 2003), in which a function is seen as the fixed relationship between the members of two sets. From this perspective, $y = f(x)$ represents $y$ as a function of $x$, in which each value of $x$ is associated with a unique value of $y$ (Farenga & Ness, 2005). This static view underlies the typical treatment of functions in school mathematics, and it is not difficult to see how students may struggle to transition from a repeated-multiplication understanding of exponentiation to a correspondence understanding, particularly beyond the domain of natural numbers.

Smith and Confrey (Smith, 2003; Smith & Confrey, 1994) offered an alternative to the correspondence view, which they called the covariation approach to functional thinking. Here one examines a function in terms of a coordinated change of $x$- and $y$-values:

A covariation approach, on the other hand, entails being able to move operationally from $y_m$ to $y_{m+1}$ coordinating with movement from $x_m$ to $x_{m+1}$. For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table. (Confrey & Smith, 1994, p. 33)
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This is what the students described in Confrey and Smith’s 1994 article did when they calculated the ratios between succeeding \( y \)-values for a constant unit-change in the \( x \)-values in tables of exponential data. Confrey and Smith argue that splitting, juxtaposed with covariation, can provide the basis for the construction of an exponential function. Exponentiation is simply repeated splitting, just as multiplication is repeated addition.

**Covariation for Exponential Growth**

Castilow-Garsow (2012, this volume) describes covariation as the imagining of two quantities changing together; students imagine how one variable changes while imagining changes in the other. Relying on situations that involve quantities that students can make sense of, manipulate, experiment with, and investigate can foster their abilities to reason flexibly about dynamically changing events (Carlson & Oehrtman, 2005). An approach that relies on imagining co-varying quantities may be especially useful in helping students understand exponential growth, as this view is strongly tied to how students think about contexts involving multiplicative relationships (Davis, 2009). Thompson (2008) argues that a defining characteristic of exponential functions is the notion that the rate at which an exponential function changes with respect to its argument is proportional to the value of the function at that argument. Approaches that emphasize this concept could help students make strong connections between the change in \( x \)-values and the corresponding change in \( y \)-values, developing the understanding that the value of \( f(x + \Delta x)/f(x) \) is dependent on \( \Delta x \) (Thompson, 2008). So, for instance, for a repeated doubling function \( f(x) = 2^x \), a covariation approach could help students coordinate (additive) changes in \( x \) values with (multiplicative) changes in \( y \)-values to understand that the constant multiplicative rate of change for \( \Delta x = 1 \) would be 2, for \( \Delta x = 2 \) would be \( 2^2 \), or 4, for \( \Delta x = 3 \) would be 8, and for \( \Delta x = \frac{1}{2} \) would be \( 2^{-1} \).

One study (Green, 2008) did take a rate-of-change approach to helping students construct exponential growth and found that expanding the concept of rate of change to include percent changes helped students understand the meanings of the parameters of exponential functions. In another study with two high-school students, Castillo-Garsow (2012) found that a focus on reasoning covariationally about financial modeling tasks fostered different solutions to a differential equation based on either a discrete or continuous understanding of change.

We were interested in developing a situation in which the notion of proportional rate of change would arise naturally. We have found that adopting a rate of change perspective can be accessible even for beginning algebra students in middle school, particularly if they have opportunities to explore situations that encourage students to construct meaningful relationships between quantities (Ellis, 2007, 2011a, 2011b). Given the age group of our students, we aimed to develop a context with co-varying quantities that satisfied three requirements. First, students should be able to visually observe the quantities changing together. Second, students should have a way to easily measure and record the values of both quantities as they covaried. Third, the quantities should vary in a continuous rather than discrete manner. We will describe how we implemented each of these criteria in the sections below.

**Continuous Variation**

In order to discuss continuous variation, it is helpful first to address the ideas of chunky reasoning and smooth reasoning (Castilow-Garsow, 2012). Castilow-Garsow describes chunky reasoning in the following manner: a student imagines that a change occurs in completed “chunks”, after a certain amount of time has passed, such as a day or a week. The student does not imagine that change occurs within the chunk unless she can re-conceptualize the change to a smaller chunk size, such as chopping a week into seven days, with each day having its own completed change. Castilow-Garsow explains that “Chunky thinking is inherently discrete. It remains an open ques-
tion whether or not continuous understanding can be built from pure chunky thinking” (p. 11).

In contrast, when reasoning smoothly, a student imagines a quantity changing in the present tense; one can map from one’s own current experiential time to a time period in the mathematical context without needing to resort to convenient units of time. “Smooth thinking, in contrast, is inherently continuous. By imagining change in progress, change is subjected to that person’s understanding of change in the physical universe” (Castillows-Garsow, 2012, p. 11). Castillow-Garsow suggests that the continuous nature of smooth reasoning is critical for understanding exponential growth.

Continuous quantitative reasoning then becomes a “repeated process of imagining the smooth change in progress of a quantity over an interval, followed by an actual or imagined numerical measurement of the quantity at the end of each interval” (Castillows-Garsow, 2012, p. 18). Thompson (2011) similarly describes the concept of continuous variation, in which every smooth change in progress is imagined to be composed of smaller chunks (giving numerical values), and every small chunk within the change in progress is thought of as being itself covered by a smooth change in progress. In this manner, a student achieves infinite precision by alternating smooth and chunky thinking by chopping the interval of variation into finer and finer chunks. If continuous reasoning relies on smooth and chunky thinking in this manner, then education targeting continuous quantitative reasoning should focus on developing smooth reasoning skills (Castillows-Garsow, 2012). For this reason we endeavored to develop a scenario in which students would have the opportunity to engage in continuous quantitative reasoning when imagining two quantities co-varying exponentially.

**The Jactus: Building Exponential Growth by Reasoning with Continuously Covarying Quantities**

Building on the principles enacted in our linear and quadratic teaching experiments (Ellis, 2007; 2011), we set out to develop a context in which students could explore two continuously covarying quantities; we wanted to avoid discrete situations such as the chessboard problem. Our intention was to develop a context that would be understandable to a middle-school population. We settled on a scenario in which a plant called the Jactus grew by doubling its initial height every week. As we will discuss later, the choice of a week as the time frame for the plant to double was deliberate. Students explored the growing Jactus plant by comparing its height to time via a specially designed GeoGebra script (Figure 1).

![Figure 1: Screenshot of the GeoGebra Script for a Doubling Jactus](image)
Students could manipulate the image of the Jactus plant by dragging its base with the mouse. As they did so, the plant would continuously increase or decrease in height as it moved along the time axis. Over time, we changed the growth factor of the Jactus to values other than 2, the initial height to values other than 1 inch, and the amount of time to double to values other than 1 week.

We recognize that situations do not imply reasoning; continuous problem situations do not necessarily mean that continuous reasoning will occur. As Castillo-Garsow (2012) demonstrated in his example with Alice, Bob, and Carol’s solutions to the same problem, a student’s reasoning may not necessarily correspond to the co-varying quantities in the situation. However, we believe that a context with smoothly co-varying quantities could afford the possibility of continuous reasoning in a way that a discrete situation would not. It is possible to imagine a plant that is somewhere between 1 inch tall when it starts growing and 2 inches tall after the first week. It is more difficult to imagine, for instance, a ruba coin that is in the middle of becoming two ruba coins.

**Encouraging Coordination**

We hypothesize that significant mathematical learning can take place as a result of students’ engagement in appropriately selected and sequenced mathematical tasks. If individuals have the capacity to learn through their mathematical activity, the possibility exists to engineer a sequence of tasks that promotes the learning of individuals through their engagement in such a task sequence. (Simon et al., 2010, p. 72)

We set out to develop a set of sequenced mathematical tasks by first identifying the ways of thinking that we desired for our students as an outcome. Carlson and Oehrtman’s covariation framework (2005) informed our thinking for the initial design of tasks. Their framework emerged to describe the reasoning involved in the meaningful representation and interpretation of graphical models in calculus, but it has still served useful to guide our thinking about what we wanted middle school students to understand about exponential growth. Carlson and Oehrtman described five mental actions, the first four being relevant to middle school students’ reasoning with exponential functions (the fifth addresses instantaneous rates of change).

The first mental action identified by Carlson and Oehrtman is to **coordinate the dependence of one variable on another variable**. In the Jactus context, we developed activities to help students understand that the height of the Jactus plant depends on the amount of time that it had been growing. Students’ interaction with the GeoGebra scripts familiarized them with this dependence relationship. We then designed tasks in which students discussed the variables that could contribute to the Jactus’ growth, drew pictures of the growing Jactus, and devised methods for keeping track of the plant’s growth over time.

The second mental action is **coordinating the direction of change of one variable with changes in the other variable**. We wanted students to understand that as time increases, the plant grows taller, and as time decreases, the plant grows shorter. Our activities with drawing pictures, interacting with the GeoGebra script, and identifying relationships between how much time had passed and how tall the plant had grown encouraged this coordination.

The third mental action is **coordinating the amount of change of one variable with changes in the other variable**. Students should initially understand that the growth in height is determined multiplicatively rather than additively. We wanted students to understand that repeated multiplication of the height, such as doubling, occurred every week. Early activities aimed at this idea required students to create drawings. For instance, given a picture of a 1-inch Jactus plant at 0 weeks, students drew the height of the plant at 1, 2, 3, and 4 weeks. Later, students
created drawings in which they had to skip weeks, for instance, drawing the height of the plant at week 3 given a picture of the plant at week 0. In order to highlight the fact that the growth rate is the same for any same $\Delta x$, we also asked problems such as the following, in which the height value is unknown (imagine that a student has already determined that a particular Jactus plant triples each week): “Say you go on vacation for 1 week. How much taller will the Jactus be when you return?” We combined these tasks with interpolation problems, far prediction problems, and comparison problems across different plants with different growth rates and initial heights in order to encourage facility with coordinating the amount of change of height with the amount of change of time.

Carlson and Oehrtman’s fourth mental action is coordinating the average rate-of-change of the function with uniform increments of change in the input variable. In this case, we focused instead on coordinating multiplicative comparisons of $y$-values with additive comparisons of $x$-values. We wanted students to understand that a) the ratio of $f(x_i)$ to $f(x_j)$ will always be the same for any same $\Delta x = x_j - x_i$, and b) the value of $f(x + \Delta x)/f(x)$ is dependent on $\Delta x$. We designed tasks that introduced tables of exponential data in which the week values were not uniformly distributed and asked students to find missing week values and missing height values. We also used these table representations when asking students to determine an unknown growth factor. Once students were already coordinating ratios of successive $y$-values to determine growth factors, we attempted to make the value of $\Delta x$ explicit by emphasizing different $\Delta x$ values, such as in the task in Figure 2:

![Image of a table task]

**Figure 2:** Non-uniform Table Task

**Encouraging Smooth Reasoning**

Another conceptual goal was that students coordinate the ratio of $f(x_j)$ to $f(x_i)$ for values of $\Delta x$ that were less than 1, ultimately being able to imagine this coordination for arbitrarily small increments in $x$. Weber (2002) argues that students should, in time, be able to generalize their understanding of repeated multiplication to make sense of what it means to be “the product of $x$ factors of $a$” when $x$ is not a positive integer. The Jactus context could allow for a meaningful interpretation of this idea when students begin to contemplate how much the plant would grow
in one day, for instance, given that it doubles in one week. We designed tasks in which students had to determine how tall a plant would be at non whole-number time values such as half a week or two and a half weeks. We then asked students to determine how much a plant would grow for many different time periods, such as for 10 weeks, for one day, or for 1/10th of a week.

**The Teaching Experiment**

We recruited three eighth-grade students (ages 13-14), Uditi, Jill, and Laura (all pseudonyms). Jill and Laura were enrolled in eighth-grade mathematics, and Uditi was enrolled in an eighth-grade algebra course. None of the students had encountered exponential functions in their mathematics classes at the time of the study. The students participated in a 12-day teaching experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000) over the course of three weeks, in which the first author was the teacher-researcher. Two project members familiar with the teaching-experiment methodology observed and videotaped each teaching session. Each session lasted approximately 1 hour. The project team met daily after each session to debrief and discuss the events that transpired during the session. For the purposes of this paper, we present an analysis of the students’ conceptual development over the course of the teaching experiment, with an emphasis on three shifts in understanding.

Before embarking on the teaching experiment, the project team developed a progression of tasks according to the design principles described above. We also created a hypothetical learning trajectory (Simon, 1995), organizing a sequence of tasks that would foster students’ understanding of exponential growth. However, the teaching experiment model demanded a flexibility that meant the initial sequence of tasks served only as a rough model for instruction. We engaged in iterative cycles of intervention, assessment and model building of students’ thinking, and revision of future tasks on an ongoing basis. In this manner we developed, during each teaching session, enhanced hypotheses of the students’ understanding based on the previous cycle (Simon et. al, 2010).

**Data Analysis**

We employed retrospective analysis (Simon et. al, 2010) in order to characterize students’ changing conceptions throughout the course of the teaching experiment. We relied on the initial learning trajectory as a source of preliminary codes for the data, coding students’ talk, gestures, actions, and responses to tasks as evidence of understanding at various stages on the trajectory. The act of coding students’ understanding also produced emergent codes, which then altered the initial learning trajectory to account for the events that occurred during the teaching experiment. The codes and the trajectory therefore evolved simultaneously in a cyclical manner until the trajectory stabilized to reflect the final set of codes identifying students’ conceptions. Although the presentation of the final learning trajectory is beyond the scope of this paper, we will present a portion of the trajectory accounting for three major shifts in students’ conceptual understanding of exponential growth.

Two members of the research team initially coded the entire data corpus independently. During this process they met weekly with the entire project team in order to discuss boundary cases and clarify and refine uncertain codes. Once this initial phase was complete, the two researchers then recoded the entire data set together by comparing every codes and discussing any differences until reaching agreement. The iterative process of coding, refining, and recoding continued until no new codes emerged and no more refinement was necessary.
From Repeated Multiplication to Constant Multiplicative Rates of Change: Three Conceptual Shifts

Early in the teaching experiment, all three students demonstrated an understanding that repeated multiplication of the height determined the manner in which the Jactus plant grew. In this analysis we present evidence of three major conceptual shifts that marked the students’ transitions from understanding exponential growth as repeated multiplication to coordinating multiplicative ratios of height values with additive differences for the time values. We will address each of the shifts (Figure 3) in turn.

| Shift 1: From Repeated Multiplication to Coordinating x and y | Students shift from only attending to the repeated multiplication of the y-values to coordinating repeated multiplication with changes in x-values. |
| Shift 2: From Coordinating x and y to Coordinated Constant Ratios | Students shift to explicitly coordinating the ratio of \( f(x_2) \) to \( f(x_1) \) for corresponding \( x \)-values for any \( \Delta x \geq 1 \). |
| Shift 3: Generalizing to non-natural exponents | Students shift to explicitly coordinating the ratio of \( f(x_2) \) to \( f(x_1) \) for corresponding \( x \)-values for any value of \( \Delta x \), including values in which \( \Delta x < 1 \). |

*Figure 3: Three Conceptual Shifts in Creating Multiplicative Rates of Change*

After exploring the GeoGebra script, the students began to quickly identify repeated multiplication as the mechanism that determined the manner in which the Jactus grew. For instance, the students encountered a situation in which the Jactus was 1 inch tall when it began to grow, and it grew by quadrupling its height every week. Uditi described its growth this way: “They’re all going up by like times 4, like, 16 times 4 is 64 and then 64 times 4 and then 64 times 4…then times 4 that’s 1024.” Explaining how she would determine the height of the plant at 7 weeks, Uditi stated, “Four times 4 is 16, 16 times 4 is 256 then 4…1024 times 4, 4096 times 4 and then it’s 16,384.” Absent from Uditi’s language is an explicit coordination of repeated multiplication of the height of the Jactus with the amount of time it took the Jactus to quadruple. The first shift we address is the one in which the students began to attend to corresponding time values when considering the height of the Jactus.

**Shift 1: From Repeated Multiplication to Coordinating x and y**

On Day 2 of the teaching experiment the students encountered a far prediction question: Given a Jactus that doubled every week and began growing when it was 1 inch tall, how tall would it be at 30 weeks? Laura’s response indicated an understanding of repeated doubling, and an attention to the corresponding number of weeks began to make its way into her language: “You can do like, so here’s 8, and then the next is 16 and I guess like I said yesterday it goes up, if that’s the rate, and if you minus it you’ll see that…for 32 you would do 32 times 2, and then you have the result for the week, for week 6 and then you just keep going like that.” At this point, there is little evidence of explicit coordination of repeated doubling of the height per a 1-week increase in time. In order to find the value of the plant’s height at week 30, Laura constructed a table from week 6 all the way to week 30 because she had no way to truncate the repeated multiplication process (Figure 4), which resulted in calculation errors.
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Figure 4: Laura’s table for determining the Jactus’ height at week 30

In an attempt to highlight the coordination of the plant’s height with weeks, the teacher-researcher (TR) introduced a task that required the students to draw the plant’s height after 1 week and after 3 weeks. After the students produced their drawings, the teacher-researcher asked them to think about Week 4:

TR: So what would happen the next week and week 4?

Uditi: It would be more bigger.

TR: It would be more bigger?

Jill: It would be double, it’s…

TR: Oh, it would be double. *Double what?*

Jill: Well, the next week would be double the last week.

Jill’s drawing indicated the beginning of her coordination of the plant’s height with the number of weeks; both are present in her picture (Figure 5). Although it appears that Jill’s image for Week 3 was of a plant twice as tall as Week 1, she labeled the Jactus as 8 inches tall:

Figure 5: Jill’s drawing of the growing Jactus
TR: So then what would week 5 do?
All: Double that.
Laura: It would double the week before.

By the fourth day, the students could explicitly coordinate the height of the plant with the week number, but they could not coordinate the growth in height for multiple-week jumps. For instance, Laura worked with a table of exponential values in which she had to find the plant’s height for week 10 and week 13 (Figure 6):

<table>
<thead>
<tr>
<th>Weeks</th>
<th>Inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>24</td>
</tr>
<tr>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td>96</td>
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<tr>
<td>6</td>
<td>192</td>
</tr>
<tr>
<td>7</td>
<td>384</td>
</tr>
<tr>
<td>8</td>
<td>768</td>
</tr>
<tr>
<td>9</td>
<td>1,536</td>
</tr>
<tr>
<td>10</td>
<td>3,072</td>
</tr>
<tr>
<td>13</td>
<td>8,4576</td>
</tr>
</tbody>
</table>

*Figure 6: Laura’s table of week / height values*

Laura was able to coordinate the growth in inches with the number of weeks the plant had been growing by filling in the gaps in the table. Her language reflected an explicit attention to weeks: “For 4 (weeks) I got 48 (inches), for 6 I got 192, for 7 I got 384, for 9 I got 1,536, for 10 I got 3072 and for 13 I got 24,576.” Laura could only double the previous week’s height to find the next week’s height, and she did not yet understand that this process could be truncated, for instance, that she could find the Jactus’ height at week 13 by taking the height at week 10 and multiplying it by 8, or $2^3$ for a jump of 3 weeks.

**Shift 2: From Coordinating x and y to Coordinated Constant Ratios**

On the fifth day Jill and Uditì encountered a non-uniform table of exponential data (Figure 7) (the function for these data is $f(x) = 0.1(2^x)$). Laura was absent on this day, and her attendance became more sporadic than Uditì and Jill’s for the remainder of the sessions.
Jill’s response demonstrates the beginning of a coordination of the change in the height of the plant with the change in weeks. She took the ratio of the height at week 8 to the height at week 0, dividing 25.6 inches by 0.1 inches, explaining, “I did 25.6 divided by 0.1 and that was 256.” Jill also knew that she had to account for the 8-week difference. However, rather than recognizing that 256 represented a growth factor repeatedly multiplied 8 times (e.g., \( b^8 = 256 \)), Jill divided by 8: “I just divided that by 8 because there was 8 weeks and I got 32, but I don’t really know why I did that, I just kind of did it.”

The teacher-researcher provided Jill with a new problem that contained only two data points (Figure 8), and asked Jill to determine how the plant was growing.

At this point Jill began to think about the gap in weeks as representing how many times repeated multiplication must occur: “I tried 3 times 3 and it was the 16 week number, and so then I figured out if I did the 14 week number times 9 it would give me this number.”

On the same day, Uditii demonstrated a similar emerging coordination between the multiplicative growth in height and multiple weeks, but did not generalize her understanding. In another non-uniform table of exponential data (for which the underlying function was \( f(x) = 0.1(4^x) \)), Uditii took the ratio of the first two height values (Figure 9)
Figure 9: Uditi’s work on a non-uniform table of exponential data

Uditi divided 1.6 inches by 0.1 inches to get 16, and she knew that this meant she had to find a number that she could multiply twice to get 16. Given her limited knowledge of algebraic manipulation with exponents, Uditi had to rely on a guess and check method: “I found out that the difference was 16 and then I just tried all these different numbers. I tried to multiply and then multiply it again the same number.” In this manner she discovered that the growth factor was 4. At this stage, however, Uditi was not yet able to generalize her reasoning to any multi-week gap; for instance, she was unable to explain that the plant would grow \(4 \times 4 \times 4\) times, or \(4^3\) times, between week 2 and week 5 because there was a 3-week gap.

On Day 7 of the teaching experiment the teacher-researcher gave the students two data points with a 5-week gap in order to encourage the coordination of the multiplicative growth in height with the additive growth in time. For this task, the Jactus was 3,355.443.2 inches tall on week 24, and it was 107,374.182.4 inches tall on week 29. The teacher-researcher asked the students to determine how the Jactus was growing (e.g., whether it was doubling, tripling, etc.) and how tall it was at week 0. Jill took the ratio of the two height values and found it to be 32, and then took the difference of the two week values. She then wrote, “ \(\cdot \cdot \cdot \cdot \cdot \)”, searching for a number that she could multiply by itself 5 times in order to yield 32. By guessing and checking, Jill determined that the growth factor should be 2 (Figure 10).

Figure 10: Jill’s solution to the two-data-point problem
Uditi’s response was similar to Jill’s, and once she found that the growth factor was 2, she used this information to determine the height at Week 0. Uditi wrote “\( y \times 2^{24}\)”, and then guessed and checked in order to find the value that would make her expression equal to the height at 24 weeks, 3,355,443.2 inches. Uditi found the missing value to be 0.2, so then she produced the equation “\( y = 0.20 \times 2^{2} \)”. Uditi and Jill also appeared to understand that the ratio of height values would remain constant for any same-number increase in weeks. For instance, after determining that one tripling Jactus plant grew 8 times as tall from Week 3 to Week 6, both students immediately predicted that the plant would be 8 times taller at Week 103 than it would be at Week 100.

**Shift 3: Generalizing to Non-Natural Exponents**

Our intention was to foster a coordination between the multiplicative growth of \( y \)-values and the additive growth of \( x \)-values even for cases in which the difference between \( x \)-values was less than 1. Our hope was that students would be able to imagine this coordination for arbitrarily small increments in \( x \), which would provide a way to understand \( b^a \) even when \( a \) is not a whole number. One way to introduce this idea was to ask the students to think about how tall the plant would be in the middle of a week; for instance, if a plant doubles between Week 1 and Week 2, how tall will it be at Week 1.5?

**Estimation and Reversion to Linear Thinking**

Early in the teaching experiment, the students struggled to make estimates for how tall the plant would be in between whole week values. For instance, on the third day of the teaching experiment the students worked with a Jactus plant that was 1 inch tall at 0 weeks and doubled each week. The teacher-researcher asked Uditi how tall the plant would be at 1.5 weeks and she guessed that it should be “a little more than three.” Uditi explained, “It increases a little more every time it goes that way [gesturing to successive weeks], so in between them is three so it’s going to increase a little more, like comes to point 5.” Uditi had a qualitative (Behr et al., 1992) understanding of the plant’s growth, in that she could reason about the direction of change in the plant’s height with relying on calculations. Uditi knew that the manner in which the plant increased from one week to the next was not linear. Because the plant was 2 inches tall at 1 week and 4 inches tall at 2 weeks, Uditi guessed that it would be slightly more than 3 inches tall halfway between 1 and 2 weeks, but she was unable to determine the precise amount of growth. Although her estimate is incorrect, it is notable that Uditi did not revert to linear interpolation when trying to determine the plant’s height at 1.5 weeks.

One week into the teaching experiment the students revisited the same doubling Jactus, but this time the teacher-researcher asked the students to think about the plant’s height every day between 0 weeks and 1 week. Laura and Jill both reverted to linear interpolation, stating that the plant would be 1.1 inches tall after 1 day, 1.2 inches tall after 2 days, and so forth. Uditi, however, relied on her ability to coordinate multiplicative growth in the plant’s height with additive growth in weeks. She took the ratio of the plant’s height at 1 week (2 inches) to the plant’s height at 0 weeks (1 inch), and then wrote “\( x \times x \times x \times x \times x = 2 \)”. At this point Uditi became stuck, because she was unable to find a number that she could multiply by itself seven times to result in 2.

**Reliance on Equations to Assist with Interpolation**

The next day the studentsvisited the same doubling plant with a table that provided its heights for different decimals between Week 0 and Week 2 (Figure 11).
Laura knew that the ratio of heights for any 0.25-week difference would be the same, so she took the ratio of the plant’s height at week 0.75 and divided it by the plant’s height at week 0.25, which was 1.189. She then wrote “goes up by 1.189207115” and concluded that 1.189 would be the plant’s height at Week 0.25. Although Laura’s “goes up by” language suggests the possibility of an image of an additive difference of 1.189 between successive quarter weeks, later Laura determined the height for Week 3.5 by using appropriate multiplicative reasoning, multiplying the value of the height at Week 3 by 1.189 twice.

In contrast, Uditi relied on her ability to create an equation in order to determine the missing height at 0.25 weeks. Uditi wrote “1 × 1.189” and then used a guess and check method to find the missing value that would yield the plant’s height at 0.5 weeks. She easily determined that the missing value was 2, and then checked the correctness of her equation with other values in the table, ultimately declaring that the plant doubled each week. Uditi then wrote “Height = 1×2^{\text{week}}” and substituted 0.25 for the exponent in order to find the missing value.

Coordinating the Ratio of Height Values for Δx with $b^{\Delta x}$
In order to encourage Uditi to coordinate the plant’s growth with parts of weeks rather than relying on an equation, the teacher-researcher introduced a problem that only had two data points (Figure 12), asking the students to figure out how the plant was growing.

**Figure 11:** Table of values for doubling Jactus

<table>
<thead>
<tr>
<th>Week</th>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1&quot;</td>
</tr>
<tr>
<td>0.25</td>
<td>??</td>
</tr>
<tr>
<td>0.5</td>
<td>1.41421356237310&quot;</td>
</tr>
<tr>
<td>0.75</td>
<td>1.68179283050743&quot;</td>
</tr>
<tr>
<td>1</td>
<td>2&quot;</td>
</tr>
<tr>
<td>1.25</td>
<td>2.37841423000544</td>
</tr>
<tr>
<td>1.5</td>
<td>2.82842712474619</td>
</tr>
</tbody>
</table>

**Figure 12:** Two data points from the function $f(x) = 3^x$
QUANTIFYING EXPONENTIAL GROWTH

Jill and Laura both took the ratio of the two height values and stated that the plant would get “1.116123172” times as big each tenth of a week, but neither student could use this information to determine how much the plant would grow in 1 week. Uditi also took the ratio of the two height values, which she rounded to 1.12. In order to determine how big the plant would grow every week, she then once again relied on an equation, writing “__ × 1.12 = 1.12”. Uditi indicated that the first blank represented the “starting number”, or the plant’s height at Week 0, and the second blank represented the unknown weekly growth factor of the plant. Although Uditi’s approach was unexpected, we found it notable because she equated the ratio of the y-values, 1.12, to $a^b$ (for a function $f(x) = ab^x$). This represents a key understanding, that the ratio of height values for a given $\Delta x$ can be expressed as $b^\Delta x$. At this point it was not clear how Uditi might generalize this understanding, and there was little evidence to determine whether Uditi also realized that this ratio will always be the same for any same $\Delta x$.

Uditi knew that the plant increased in height by a factor of approximately 1.12 for 1/10th of a week. Rather than taking this value to the tenth power in order to determine how much the plant would grow in one week, Uditi simply used a guess and check method with her equation to determine that $3^{10} \approx 1.12$. She concluded that the plant tripled every week. The teacher-researcher asked Uditi how the plant grows every 2 weeks and every 3 weeks, and Uditi concluded that every 2 weeks the plant would grow 9 times as big and every 3 weeks it would grow 27 times as big. She explained, “Because $3 \times 3 = 9$ and $3 \times 3 \times 3 = 27.” The teacher-researcher decided to ask Uditi about larger numbers:

TR: What if I gave you really big numbers, like Week 100 and Week 101? Is this number, whatever it’s going to be, still going to be 3 times as big the next week?
Uditi: (Nods).
TR: Even all the way up here?
Uditi: (Nods).
TR: How come?
Uditi: Because, the equation says it grows 3 times every week.
TR: Okay, and if it’s going 9 every 2 weeks are you confident that it will be that way for any two weeks?
Uditi: (Nods).
TR: How come?
Uditi: Because, 3 times 3 times 3 is 9.

Based on Uditi’s responses, it was unclear what she truly thought; Uditi could have possibly have just been responding positively to the teacher-researcher’s questions. Therefore the teacher-researcher gave her another problem with only two data points. This plant was 64 inches tall at 3 weeks, and 68,719,476,736 inches tall at 18 weeks. The students had to figure out how the plant was growing every week, every 3 weeks, and every ½ week. Both Uditi and Laura determined that the plant grew 4 times as large each week by taking the ratio of the two height values and then determining what number to the 15th power (the difference between 18 weeks and 3 weeks) yielded that result. Both then concluded that the plant would grow 64 times as large every 3 weeks. Uditi then explained that the plant would grow 2 times as large every ½ week, reasoning that at 3 weeks the plant was 64 inches tall, at 3.5 weeks it was double that, 128 inches tall, and at 4 weeks it doubled again to 256 inches tall (Figure 13).
The teacher-researcher then asked Uditi how the plant would grow every $x$ weeks, and she responded, “It grows by 4 to the $x$.”

### Generalizing to Fractional Exponents

On the last day of the teaching experiment, the teacher-researcher gave the students a task designed to determine whether they understood that ratio of $f(x_2)$ to $f(x_1)$ will always be the same for any same $\Delta x = x_2 - x_1$, even when $\Delta x$ is a fraction (Figure 14):

![Figure 14: Tripling Jactus Table](image)

We were curious whether the students would attempt to make a calculation, or whether they would know that the plant would grow 9 times as big regardless of its size at any given week. Laura did not recognize this and attempted to calculate an answer. Jill and Uditi both immediately said the plant would become 9 times as big. Jill explained, “I noticed that 1 times 9 is 9 and 2, or, 9 times 9 is 81, so every 2 weeks it is going up 9 times.” The teacher-researcher then asked the students how much the plant would grow from Week 155 to Week 160, and Uditi wrote “$3^5 = 243$”, explaining that this was a way to represent “three times three times three times three times three.” Laura and Jill were then able to use Uditi’s reasoning to correctly determine how much larger the plant would grow in 10 weeks.

The teacher-researcher then asked the students to determine how much the plant would grow in 1 day, or $1/7$th of a week. Uditi wrote “$3^{14} = 1.17$”, explaining, “It’s 1 divided by 7 because it only shows the result for 1 week on the table, and there are 7 days in a week. So I divided 1 week into 7 parts, which represents 1 day each and it’s .14 of a week.” In this manner Uditi was able to make sense of a non-whole number exponent and generalize her coordination of the ratio of height values with growth in time to $\Delta x$ values less than 1.
Discussion

The students began the teaching experiment with an understanding of exponential growth as repeated multiplication, but they did not explicitly coordinate repeated multiplication of height with growth in time. Over time, the students began to coordinate the growth of the height of the Jactus with time, first implicitly by thinking about the plant as doubling “every time”, and then explicitly as they had to negotiate non-uniform tables of values. The students then began to truncate the process of repeated multiplication, coordinating increases in height with multi-week time spans. Initially they did this linearly, by dividing the ratio of height values by $\Delta x$, but ultimately they shifted to coordinating the ratio of height values with $b^{\Delta x}$ for the growth factor $b$. The students realized that this relationship would hold for any same $\Delta x$; at first, they understood this only for $\Delta x$ values greater than 1, representing multi-week jumps, but ultimately one student, Uditi, was able to coordinate for $\Delta x$ values less than 1, making use of fractional exponents.

The students’ ability to coordinate the ratio of height values with the additive difference in time values played a significant role in supporting their ability to develop algebraic representations of the plant’s growth. This was evident, for example, in Uditi’s development of the equation $0.2 \times 2^t$, in which she had determined the growth factor 2 by taking the ratio of two height values 5 weeks apart and then determining the number $b$ such that $b^5$ was equal to that ratio. In general, the students’ covariational thinking preceded their ability to develop correspondence rules of the form $y = f(x)$, which reflects Smith and Confrey’s (Smith, 2003; Smith & Confrey, 1994) assertion that students typically approach functional relationships from a covariational perspective first.

One reason why we focused on the middle school population for this study was because students in our participating schools encounter exponential growth formally in their mathematics classrooms in eighth grade. We were interested in exploring students’ evolving conceptual development as they encountered exponential situations for the first time, which necessitated a younger participant group. However, this also resulted in some challenges and constraints in the types of problems and tasks we were able to design. Enabling the students to physically manipulate and visually observe the growing plant with GeoGebra supported a qualitative experience of the nature of exponential growth, but it limited the growth factors we could use. A growth factor larger than 4 resulted in numbers too large for GeoGebra and the students’ scientific calculators, so they only had opportunities to explore plants that doubled, tripled, or quadrupled.

In addition, we typically constrained the growth factor to whole numbers because our participants did not have access to sophisticated algebraic manipulation abilities or the notion of logarithms. This meant that they were limited to guess and check methods for determining the growth factor for mystery plants. For instance, the students often determined the growth factor by taking the ratio of two height values a certain number of weeks apart. Imagine a situation in which the growth factor is approximately 97.66 for a time period of 5 weeks. A high school or college student could write the equation $b^5 = 97.66$ and then solve for $b$, calculating $97.66^{1/5}$ in order to determine that the mystery growth factor is 2.5. Because the middle school students did not possess this degree of facility with algebraic manipulation, they instead had to guess and check to determine a number they could multiply by itself 5 times that would equal 97.66. If the growth factor was something other than a whole number, the guess and check method was typically too time consuming to allow for any significant progress within the constraints of the teaching experiment.

Despite these limitations, however, the results presented in this paper offer a proof of concept that even with their relative lack of algebraic sophistication, middle school students can engage in an impressive degree of coordination of co-varying quantities when exploring exponential growth. In addition, we have presented evidence that students can generalize their understanding of expo-
nential growth to view $b^a$ as a factors of $b$ for non-natural values of $a$, as suggested in theory by Weber (2002). We contend that reasoning with continuously co-varying quantities was a critical element in constructing this particular understanding of exponentiation. The Jactus context offered a scenario in which students could begin to make meaningful sense of non-natural exponents as they imagined the height of the plant smoothly growing over time. Although we hesitate to make the definite claim that Uditi employed smooth thinking, we suspect that contexts such as the Jactus situation can support this development because they provide opportunities for students to re-imagine the nature of the plant’s growth for varying units of time.

**References**


Castillo-Garsow, C. (Submitted). *The role of various modeling perspectives in students’ learning of exponential growth.*


