Quantitative Reasoning in a Digital World: Laying the Pebbles for Future Research Frontiers

Hollylynne Stohl Lee
North Carolina State University

Introduction

In the 2010 monograph of WISDOM[e], Olive discussed a framework proposed by Olive and Makar (2009) that highlighted the interactions among technology, tasks, teacher, and students. They assert that the interactions among these four didactic variables can create opportunities for students to develop new mathematical practices and knowledge. The focus of this paper is to discuss how different types of digital tools and technology-enabled tasks can promote the development of students' quantitative reasoning practices and quantitative knowledge. My hope is to inspire the future research we need to make sense of students' quantitative reasoning using newly developed forms of interactive digital tools.

Olive previously gave examples of new media and research with Logo and robotics, on-line games environments (both single and multi-user), and face-to-face collaborative problem solving using multi-touch surfaces (see Olive, 2010, for his discussion of these). Prior research has documented the benefits of programming in Logo (and other languages). Although the ability to program Lego robots has been possible since 1998, new forms and access to robotics in schools afford opportunities that were previously not available on a wide-scale. When students are engaged in writing commands to control the actions of a robot, new practices can emerge that involve reasoning about quantities as needed to put one's creative imagination into action. Students have to make strong connections between quantitative measures such as distance and angle in order to enact a sequence of events that they imagine with their robot. Online game environments tap into students' natural love of play and can develop practices of using mathematics to achieve goals within a playful environment. While some games are created to develop skill mastery in a fun environment, others games are developed in which players have to engage in quantitative reasoning in order to achieve goals within a game (e.g., destroying monster globs by moving a laser beam along a vertical axis and aiming it so that the laser goes through as many as globs as possible with each shot; knowing how many bricks to retrieve from a mine to build a protective wall within an environment; mixing solutions correctly to add to an aquarium tank to keep the environment stable). Devlin (2011) provides an excellent vision for how games can change mathematics education.

In 2010, Olive focused his discussion on multi-touch surfaces that included interactive whiteboards and large tables that students could gather around to manipulate representations of mathematical objects. These surfaces can allow multiple users to work together on virtual tasks just as they can with a table full of concrete objects. However, in these virtual environments, the actions students can use can be drastically different. Students can not only rotate a 3D object, but they could unfold it into a net, change various properties of the object and dynamically experience these changes through touch. These experiences can afford practices of changing quantities to observe effects, and opportunities for students to observe that multiple simultaneous changes (through several students doing different manipulations at the same time) can have dramatic impact on a single object or set of objects. A practice of coordinated team work will likely emerge as students learn that to accomplish a goal everyone has to work together.

In this paper, I extend what Olive (2010) began and consider what new quantitative reasoning practices might arise with the use of new technological capabilities for exploring quantities.

Many of the examples are situated in the use of data and how tools can engage students in reasoning about quantities in the world around them. The paper is organized to highlight recent advances in digital tools, highlight research and theoretical perspectives, and lay a foundation for future research questions, concerning the use of digital tools for:

- accessing data that is meaningful to us,
- visualizing data,
- creating models and simulations of phenomena, and
- interacting with quantities using multi-touch environments and gestures.

Accessing Meaningful Data

Digital devices such as computers, tablets, and smartphones are widely available for use by individuals and in schools. The vast majority of these devices have wireless internet access, and many have built-in cameras. As of September 2012, in the U.S., about 74% of teens age 12-17 have mobile access to the internet and about 81% of adults 18 years and older report using the internet to search for information (e.g., health, hobbies, interests), find a map or directions, email, check weather or news, and engage with social networks (Pew Internet & American Life Project, 2012). Users of these devices, therefore, have easy and quick access to other digital device users, information, and data that can assist in their daily lives or be used to explore interesting questions.

Access to vast amounts of data and information opens up the possibility of developing new quantitative practices and knowledge. Chance et al. (2008) and Gould (2010) describe how easy access to large, messy, real world data afforded by technology tools can be a major benefit to the teaching of statistics. When students work with real data, rather than contrived small data sets in textbooks, they can engage more with the context of the data and the purposes behind using statistics to make sense of questions that arise from contexts. Large data set analysis have influenced new paradigms in science such as computational science and data intensive science. These are new approaches to science that supplement work done in theoretical and applied science. The potential for new quantitative practices exist with this "easy access" to large and messy sets of data.

Such new quantitative practices include developing good searching habits, learning what types of search engines to use for certain purposes, and which websites and organizations are posting trustworthy data and information. Many may not see this as a new quantitative practice. However, if the purpose of the search is to obtain quantifiable information or a measure of an attribute of interest that can be used to solve a mathematical, statistical, or scientific task, then, I propose that such search practices are a new way to reason about and with quantities. We want students to know that data and information produced from large scale surveys and research by reputable organizations and government agencies should be used when available. Additionally, if they see a statistic reported in the news, a social media post, or on a website, they should know to check the source and make a decision about the trustworthiness of the data or information reported.

One of the beautiful, and difficult, aspects of increased capacity in digital storage, transfer, and process rates, is that data sets are complex in many ways. They are often multivariate, include many cases, are presented in different formats (table or graph on website, within a PDF file, downloadable as XLS, CVS, TXT, etc.), and are often already aggregated in ways that may be more or less useful (e.g., percents or averages reported for subgroups). For example, consider two types of data: 1) a custom data set built from live births in a given year in a particular state, and 2) a large messy raw data set on automobile fuel economy. The first type of data can be accessed in a variety of ways from state or national health center statistics (e.g., North Carolina Health Data Query System, http://www.schs.state.nc.us/schs/data/query.html). The query system used in North Carolina includes an easy-to-use website to access aggregated or disaggregated reports of the number of births in each county within the state (see Figure 1). The data is returned in either a table within a webpage or a downloadable spreadsheet file. Depending on the question of interest, a user needs to build their custom data set by selecting various attributes of interest and anticipating the structure of the data set that will be generated. It would also be useful to export data with several different combinations of attributes in order to better understand the structure and determine which attributes may be most useful for a certain task. For example, I have used data from this website to have students investigate whether two male birth rates from different counties are about equally likely to occur, or if one should be considered more unusual. Thus, I was not interested in variables such as birth weight and mother's age. I only wanted access to birth data on number of each gender born from each county in NC.

The second type of data is collected and posted at a government website (www.fueleconomy.gov) aimed to help consumers make informed decisions. There are many ways users can do targeted queries or searches on this website to compare vehicles. However, the data used within all these search tools is also available for download as a complete data set (Figure 2). The data sets for each year are large and messy with over 1000 cases (vehicles manufactured) and 100 attributes. To make this data useful, a user needs to understand the structure of the data set, know what the attributes are and how each attribute is being measured. There may only be a few attributes of interest in the data for a particular question. Therefore, a teacher or student needs data reductions skills to trim the data to the attributes and cases of interest.

Figure 1. Partial view of NC Live Birth data query at http://www.schs.state.nc.us/schs/data/births/bd.cfm

Figure 2. Fuel Economy data available at http://www.fueleconomy.gov/feg/download.shtml

Gould (2010) has also highlighted that students are often surrounded by data that are meaningful to them personally, such as their song lists, phone contact lists, friend network and activity on social media sites, and logs of game play activity from web-based games. Most digital data can be exported by users. For example, Facebook allows users to download and archive personal data about their activity and friends (through Account Settings). Facebook apps created by Wolfram Alpha (http://www.wolframalpha.com/facebook/) and StatCrunch (http://www.statcrunch.com/frienddata/) use your Facebook data and return reports and graphics of various attributes. Music libraries such as iTunes typically allow for a view of song lists that can easily be copied and pasted. Figure 3 shows my song list sorted by length of song. What can you tell from this small sample of data? What questions arise from such personal data? I want to know if some artists or genres in my personal list tend to have higher song lengths than others. The longest songs in my list are heavily represented by a single artist. When I look at the entire song list, does this pattern still hold? I also wonder how strong the relationship is between song length and file size. How would these same questions be answered with someone else's song list? Could we combine song lists to do a deeper investigation? Of course, pasting the data into another application will also likely involve some data processing to ensure quantitative measures, such as time and file size, are in a numerical format that can be used for certain types of graphical displays.

✓	Name	Time ▾	Artist	Album	Genre	Kind	Size
✓	Seek Up	21:00	Dave Matthews Band	The Gorge Disc 5	Rock	MPEG audio file	19.3 MB
✓	Two Step	20:03	Dave Matthews Band	The Gorge Disc 4	Rock	MPEG audio file	18.4 MB
✓	Two Step	19:02	Dave Matthews Band	The Gorge (1 of 2)	Rock/Pop	MPEG audio file	17.5 MB
✓	Lie In Our Graves	18:37	Dave Matthews Band	The Gorge Disc 6	Rock	MPEG audio file	17.1 MB
✓	Lie In Our Graves	17:00	Dave Matthews Band	The Gorge (2 of 2)	Rock/Pop	MPEG audio file	15.6 MB
✓	Bartender	16:37	Dave Matthews Band	Weekend on the Rocks (Disc 1)	Rock/Pop	MPEG audio file	15.3 MB
✓	Bartender	16:37	Dave Matthews Band	Weekend on the Rocks (Disc 1)	Rock/Pop	MPEG audio file	15.3 MB
✓	Moon Meadow	16:34	Various Artists	Natural Encounters: From Dusk to Dawn	New Age	MPEG audio file	15.2 MB
✓	Moon Meadow	16:34	Various Artists	Natural Encounters: From Dusk to Dawn	New Age	MPEG audio file	15.2 MB
✓	Jimi Thing	15:24	Dave Matthews Band	The Gorge Disc 4	Rock	MPEG audio file	14.1 MB
✓	#41	15:21	Dave Matthews Band	Weekend On The Rocks	Rock/Pop	MPEG audio file	14.1 MB
✓	#41	15:21	Dave Matthews Band	Weekend On The Rocks	Rock/Pop	MPEG audio file	14.1 MB
✓	Two Step	14:39	Dave Matthews Band	Listener Supported (2 of 2)	Rock/Pop	MPEG audio file	13.5 MB
✓	Two Step	14:39	Dave Matthews Band	Listener Supported (2 of 2)	Rock/Pop	MPEG audio file	13.5 MB
✓	Angel	14:30	Dave Matthews Band	Live at Folsom Field, Boulder, Colorado (2 of 2)	Jam Band	MPEG audio file	13.3 MB
✓	Angel	14:30	Dave Matthews Band	Live at Folsom Field, Boulder, Colorado (2 of 2)	Jam Band	MPEG audio file	13.3 MB
✓	Jimi Thing	14:10	Dave Matthews Band	Live in Chicago 12-19-98 at the United Center (2 of 2)	Jam Bands	MPEG audio file	13 MB
✓	Jimi Thing	14:10	Dave Matthews Band	Live in Chicago 12-19-98 at the United Center	Jam Bands	MPEG audio file	13 MB
✓	Jimi Thing	14:10	Dave Matthews Band	Live in Chicago 12-19-98 at the United Center	Jam Bands	MPEG audio file	13 MB
✓	41	14:07	Dave Matthews Band	The Gorge Disc 1	Rock	MPEG audio file	13 MB
✓	Bartender	13:42	Dave Matthews Band	The Gorge Disc 1	Rock	MPEG audio file	12.6 MB
✓	Seek Up	13:29	Dave Matthews Band	Live at Red Rocks 8.15.95 (1 of 2)	Rock/Pop	MPEG audio file	12.4 MB
✓	Seek Up	13:29	Dave Matthews Band	Live at Red Rocks 8.15.95 (1 of 2)	Rock/Pop	MPEG audio file	12.4 MB
✓	Jimi Thing	13:13	Dave Matthews Band	Listener Supported (1 of 2)	Rock/Pop	MPEG audio file	12.2 MB
✓	Man of a Thousand Faces	13:10	Regina Spektor	Far [CD/DVD] Disc 1	Folk	MPEG audio file	12.1 MB

Figure 3. Personal song list, ready for easy copying and pasting into other applications.

With easy access to digital cameras and recorders, which are often integrated into mobile devices, data can also be in the form of still images or video. Images can provoke many measurement questions and help students quantify relationships or create models of their daily experiences. Consider how a photo of the San Francisco Bay bridge (Figure 4) could provoke reasoning about quantities such as length or height, a proportional relationship between the bridge and ship, and perhaps building a model of different components of the bridge. What if this image was a video instead that showed a truck driving across the bridge. How could the video be used as data to reason about how fast the truck was driving? Even though the use of images and videos have been used within mathematics and science classrooms for many years, the integrated cameras in mobile devices open up possibilities of using a single device to record and collect data, as well as use applications within the device to engage in quantitative reasoning. On an iPad, for example, two such applications are MultiMeasure for still images, and Video Physics for the analysis of video data.

Figure 4. An image as a source of data

Let's consider an example of a problem context that we will revisit throughout this paper, to highlight how a student may develop and use different quantitative practices because of access to new digital tools.

> *Suppose you have a classmate named Jane, or maybe you know a couple who are considering naming their daughter Jane. A few questions may arise in these contexts:*

- Can you think of other people named Jane or uses of Jane in pop culture media/ literature?
- Is Jane a popular name?
- Are there many people named Jane around the same age as your classmate?
- If you named your daughter Jane, would it be a popular name in her age group?
- If you met a person named Jane, what is your best guess at their likely age?

Within a class setting, students might generate people they have heard of named Jane, some may be known to them personally, but most will be famous or popular, such as: Jane Fonda, Jayne Seymour, and maybe popular literature/media characters such as Dick and Jane (book series), Jane of the Jungle (from Tarzan), G.I. Jane (movie), or Jane's Addiction (rock band). They may use a search engine to find people named Jane or inquire about Jane's age. While a search on "famous people named Jane" yields fruitful information of names, images, and demographics, a search on "Jane's age" yields little useful information (see links shown in Figure 5). In fact, I think this search is rather informative regarding the popularity of certain tasks related to age in mathematics classrooms. Surely we can pose better tasks about Jane's age!

Figure 5. Search for “Jane’s Age” yields links to math problems.

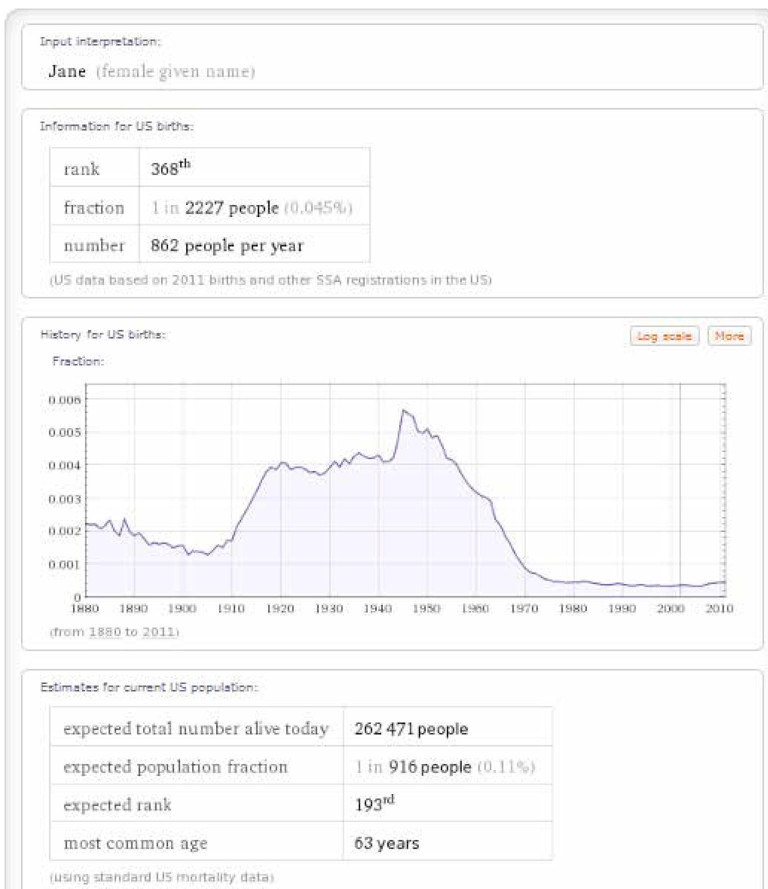

Figure 6. Wolfram Alpha’s output for a search on Jane.

Instead of using a general search engine, questions about the popularity of the name Jane and the likely age of a person named Jane seem to need a different kind of search engine, one that is data based and has computational power. Wolfram Alpha is one such search engine. Wolfram Alpha takes all forms of input and applies the computational power of Mathematica to the input differentially, based on the form of input. Entering a mathematical expression will harness the computer algebra system, while entering a word like Jane will invoke data analysis from a variety of sources to return information known about Jane. Figure 6 is the beginning of the output returned with Jane. Wolfram Alpha does not return raw data, but analyzes data from several sources (i.e., 2011 US births, Social Security Administration records, and US mortality data) and returns information about the relative frequency and popularity of the name by assigning a ranking and providing a graph of the fraction of children named Jane each year since 1880. The popularity rank of 368th, the graph illustrating the low proportion of children named Jane since 1980, and the most common age of 63 can all assist students in answering the questions posed above. This information can complement the information found through general searches that yielded information about famous people named Jane and the use of Jane in popular media. Do the references to people and media make sense with the peak shown in the graph in Figure 6 and the reported common age of 63?

Mobile digital devices with wireless connectivity to the internet may allow a new practice of “data detective work” to emerge as part of quantitative reasoning and literacy: the practice of finding, trusting, and processing data into a form that can add meaning to a task at hand. Such detective work should also lead to a habit of question posing and exploratory data analysis, hopefully with data that is personally meaningful to our students. These advanced digital tools and emerging new practices afford opportunities for future research concerning the following:

- What do students need to understand about the structure of a data set to make informed decision about the usefulness of different attributes or the best ways to transform data into useable forms with particular software?
- If teachers want to find and prepare data for their students, what understandings do they use to choose and prepare data sets? Is this a new type of knowledge needed for teaching statistics? How do we develop such knowledge?
- What aspects of data sources influence students’ trustworthiness of a source?
- Do students ask more interesting questions and engage in deeper exploratory analysis if data is more meaningful and personal to them?

Visualizing Data

Digital tools have been capable of quickly creating simple graphical displays of data for over two decades and have shifted the focus in schools away from graph construction on paper (though some of this is still done and has an important role) to graph construction with digital tools and increased attention to graphical interpretation. Even though a seemingly simple graph, Figure 7 is another part of the output from Wolfram Alpha’s search on Jane. The user is left to interpret the meaning of this curve, the area underneath, and the scale on the y-axis. If this distribution represents the current population of people named Jane, how could this distribution be used to help answer the question of a likely age of someone you meet named Jane?

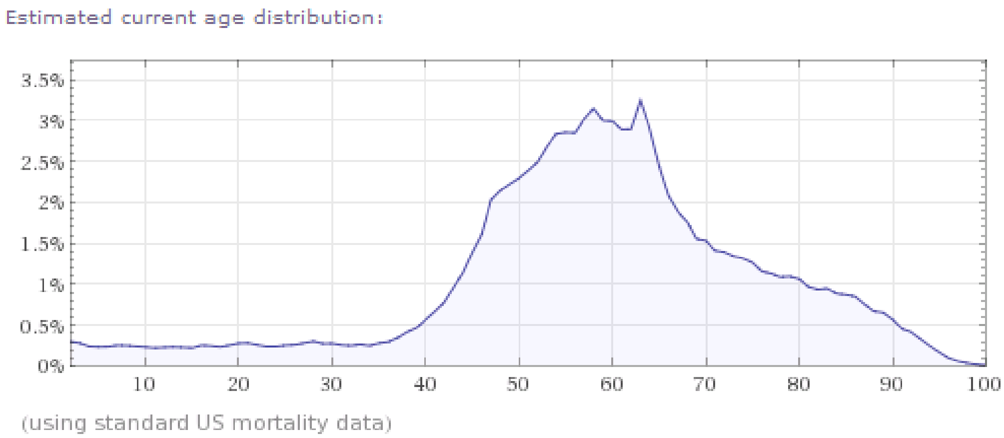

Figure 7. Distribution of ages for people named Jane in the US.

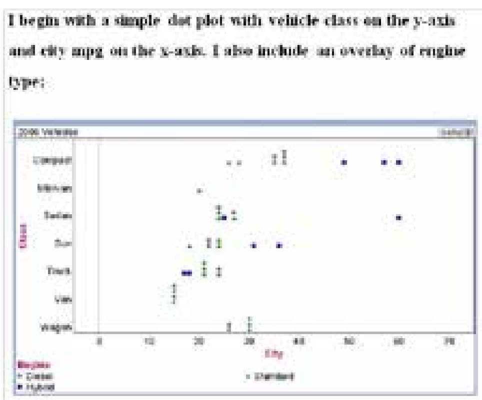

Figure 8. Graph of three attributes

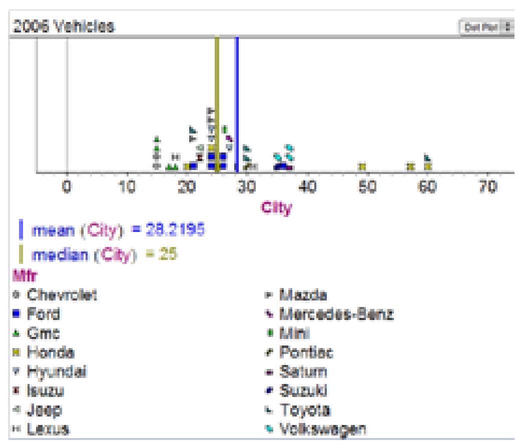

Figure 9. Graph of two attributes with measures of center marked

Much research has been done concerning students' use of graphs and their ability to reason about and with data in graphical forms (e.g., delMas, Garfield, & Ooms, 2005; Friel, Curcio, & Bright, 2001). Some early research on graph sense focused on students' ability to create and interpret standard graphical forms such as dot plots, bar charts, histograms, box plots (Bright & Friel, 1998; Pereira-Mendoza & Newfoundl, 1991). Newer technologies, occasionally used in schools, such as Fathom, TinkerPlots, and InspireData, allow for different graphical forms to be created and linked together, giving teachers and students tools that can afford new perspectives on data (e.g., Rubin, Hammerman, & Konold, 2006). In recent research by myself and colleagues (Lee, et al., 2012; Lee et al., under review), we examined how a sample of 62 prospective mathematics teachers across eight institutions used tools in Fathom and TinkerPlots to create graphs to explore questions arising from a multivariate dataset. In particular, we noticed that a high proportion (77%) of prospective teachers took advantage of new ways of augmenting a graph with additional information, such as using colored icons to represent new attributes, or displaying a measure within a graph. The augmented graphs in Figure 8 and 9, made by prospective teachers, show data from a sample of vehicles from the Fuel Economy website (shown in Figure 2). We found that teachers that created graphs like these had deeper insights into the data, which often led to posing questions to dig deeper into the data. Thus, augmenting graphs led to further exploratory data analysis.

In the past decade, many additional data visualization tools have been developed and new graphical forms, enhanced with color and animations, are being used to help illustrate and tell stories with data. Tools such as GapMinder (www.gapminder.com) and Google Public Data (www.google.com/publicdata) are free and have built-in access to large global data sets concerning economy, population, health, infrastructure, education, and technology. As shown in Figure 10, a GapMinder graph has four attributes displayed by using a horizontal axis (income per capita), vertical axis (life expectancy), color (geographic region), and relative size of the points (population total). In addition, a slider allows the addition of a fifth attribute, time, that allows the graphical display to be animated through time. More flexible data visualization tools such as Tableau Public (freely available at www.tableausoftware.com) allow for easy importing of your own data sets and many options for creating visualizations. However, most use of these tools is within industry, with some use in higher education institutions. Although these powerful data visualization tools are advocated for use within middle and secondary schools, few schools actually use these tools. For an example of students making sense of data using GapMinder, see a video posted at www.gapminder.org/for-teachers. Some researchers have noted that students seem to easily engage with multivariate data sets (e.g., Gould, 2010; Ridgway, Nicholson, & McCusker, 2008). However, we know little from empirical research about how students interpret the multivariate view of data within tools like GapMinder and even less about students' ability to meaningfully construct data visualizations that highlight important aspects of data.

Figure 10. Default graph on GapMinder

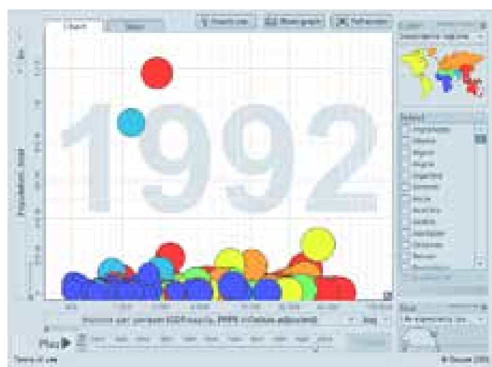

Figure 11. Changing placement of two attributes

Just because a technology tool affords a drag and drop interface for building graphs, students and teachers must make sense of the quantities represented in the data set, and should make informed decisions about how to best visually represent those quantities. For example, in Figure 11, the graph has the same variables displayed as Figure 10, but the vertical axis represents population and the size of the points represents life expectancy. With the display in Figure 11, our focus becomes the population size of dense countries such as China (red) and India (light blue), and it is hard to see relationships among the variables of the other countries, except the wide variation in income per capita, since they are clustered at the bottom and their relative sizes are hard to distinguish. However, in Figure 10, we are likely to notice the large population of

India and China as indicated by the size of the points, but we can also make sense of the trend in life expectancy as it relates to income per capita and are therefore more likely to notice the clustering of different geographic regions in the scatterplot.

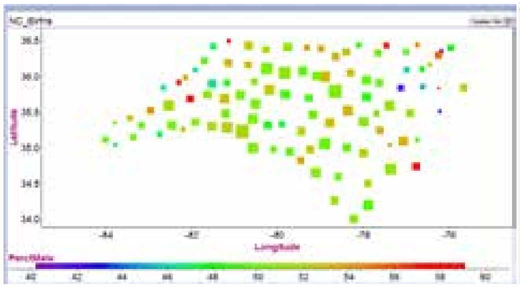

Figure 12. Graph map created in Fathom of male birth rates and total number of births in NC counties.

Figure 13. Rectangular cartogram of male birth rates and total number of births in NC counties.

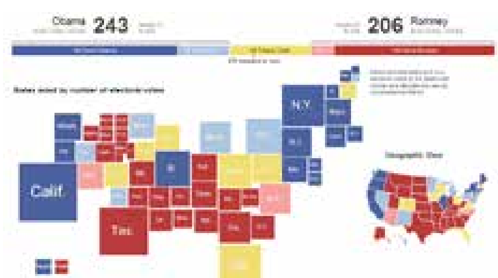

Figure 14. New York Times political visualization available at elections.nytimes.com/2012/ratings/electoral-map.

Figure 15. Network graph of Facebook friends.

Data can be visualized in other new and different ways such as using latitude and longitude data to create "graph maps," geographic maps to display an attribute for a region, rectangular cartograms, or through network maps. These newer forms of data visualization have the potential to introduce different ways of reasoning with quantitative data. Integrating statistical data with geographic data can help students to consider how social, cultural, and geographic aspects of our world may be related to various trends seen in data. For example, the graph map in Figure 12 represents the percent of males (color) and total number of births (size of squares) in counties in NC (data from query in Figure 1) and may help students notice that the very rural counties along the borders of NC are the only ones that have extreme values away from an expected 50% rate (shown in red or blue). This same data is represented in the rectangular cartogram in Figure 13 where each rectangle size represents the total number of births and the dark red or dark green represent percent of male births significantly different than the mean (shown in tan). Again, such a view challenges students to make sense of the quantities of total number of births and percent of male births as represented by a size of the rectangular region and color intensity. The same coordination of two quantities is needed for the rectangular cartogram shown in the political forecasting image in Figure 14. The network graph in Figure 15 is generated by Wolfram Alpha's analysis of my Facebook network of friends. Each node represents a friend and the lines illustrate how each friend is connected to my other friends. The colors indicate key people in my network that serve certain roles in my network of friends.

Univariate and bivariate data analysis with typical graphical representations used in schools for the past 40 or 50 years (e.g., bar charts, box plots, histograms, pie charts, scatterplots) can no longer be standard goals for quantitative and statistical literacy. Students need to reason about covariation between multiple variables and use the power of digital tools to represent quantities and measures in new ways using intuitive visual interfaces. Morton et al. (2012) describe a cycle of data inquiry and visualization with these new tools as:

> The process starts with some task or question that a knowledge worker (shown at center [of diagram in article]) seeks to gain understanding. In the first stage, the user forages for data that may contain relevant information for their analysis task. Next, they search for a visual structure that is appropriate for the data and instantiate that structure. At this point, the user interacts with the resulting visualization (*e.g.* drill down to details or roll up to summarize) to develop further insight. Once the necessary insight is obtained, the user can then make an informed decision and take action (p. 807).

Research on students' and teachers' understandings while using such visualization tools is still a new endeavor. However, based on their work with students using complex data visualization tools, Nicholson, Ridgway and McCusker (2010) hypothesize "that working with visual representations of multivariate data at an early stage would help students to develop mental models of possible relationships between multiple variables which would give them a stronger conceptual basis for considering the formal statistical analysis they will meet in courses such as psychology or geography" (p. 1). As such, new practices need to emerge in which students are able to make informed decisions about how to visualize data in complex ways and how to reason about quantities represented in complex visualizations. I wonder:

- What processes do students go through to decide how to best represent quantitative data?
- What are the implications of short term and working memory capacity on students' ability to make sense of and interpret multivariate data displays? What forms in the display are the most prominent and seem to draw attention?
- What do students need to know about quantitative attributes and measures in relation to categorical attributes such a geographic location to reason with complex data displays?
- How do early experiences with complex multivariate data sets and visualization tools affect students' abilities to make informal and formal inferences?

Creating Models and Simulations

There is often a need to quantify an aspect of phenomena in our world. This quantification process is non-trivial. If we try to create a model of some aspect of our world, it is important to make any assumptions in the model clear. The use of digital tools often requires us to make aspects of our model explicit. For example, to use Multi-Measure to explore the proportional relationship between the length of the bridge and ship in the photo, a user must define a unit of measure. If an appropriate estimate for either object is unknown, an alternative way to use measurement to compare the two objects must be developed. One approach is to measure the ship with a segment and set that length to be one unit. Then a measurement of the length of the bridge would be in ship length units. The digital tool required that I create a way to model the relationship between the two objects. If a video analysis tool, such as Video Physics (iPad) or Logger Pro (on computers) is being used to analyze the motion of an object, a model is dependent on a defined scale and the location of the origin of a coordinate plane overlayed on the video. A good example of how a physics teacher has students record and analyze videos of cars passing in front of the school to investigate a concern about speeding can be found at http://fnoschese.wordpress.com/2010/10/18/speeding-problem/.

Models can also be developed and used to make connections between data and chance. Several researchers use similar approaches in their work to emphasize the importance of making connections between theoretical or real world phenomena, a model of such phenomena, and data from the real world or generated from the model (e.g., Konold & Kazak, 2008; Lee & Lee, 2009; Lehrer, Kim & Schauble, 2007; Stohl & Tarr, 2002). Working with a variety of digital tools, these researchers have shown that modeling practices involve sense making of real world phenomena and defining attributes of the phenomena that can be measured in some way. To create a model within a digital tool requires that these attributes be defined. However, once a model is designed, the digital environments afford the collection of data from a model that can be used to test the model by comparing model-generated data to real-world data, and can also be used to answer questions about the real world phenomena that may not be possible without a data-generating model. To illustrate, let's return to the context of Jane's age.

The output from the search with Wolfram Alpha provided information that could be used to build a model to address the last question posed "if you met a person named Jane, what is her likely age?" The age distribution graph (Figure 7) provides a population density curve where the probability of choosing a person named Jane is represented as area under the curve (with the total area being one). To answer my question about the age of a person named Jane that I meet, I can image the meeting process as repeatable, where I am repeatedly meeting one of the 262,471 people named Jane in the US (Figure 6). To build this model within TinkerPlots, I am going to simplify the meeting process as one that occurs at random, where each of the 262,471 people has an equal chance of meeting me at each encounter. This is an assumption in my model and may not actually map onto how I might encounter people named Jane in my daily life. Using a curve to create a model, I set a scale from 0-100 and approximated a curve that would represent a probability distribution. This distribution represents the chance of meeting a Jane with different ages from 0-100. In my model, I want to draw (meet) one Jane at a time and record her age. I can repeat this process. Figure 16 illustrates the model and taking a sample of 500 events of meeting a person named Jane. The table and dotplot show the distribution of ages in the sample with the middle 50% marked (about age 50-70) and mean and median slightly less than 60. Students now have opportunities to reason about the quantities in the original age distribution and the model-generated data and can consider the goodness of the model.

What new practices emerge when students and teachers have access to advanced digital tools for creating models based on data they collect? One quantitative practice may be a bidirectional reasoning (Lee & Lee, 2009; Stohl & Tarr, 2002) between quantities represented in data and models. Data can assist in building models, and models can generate data. At the core of this reasoning is the issue of measurement. With such use of tools and bidirectional reasoning in place, future research can examine:

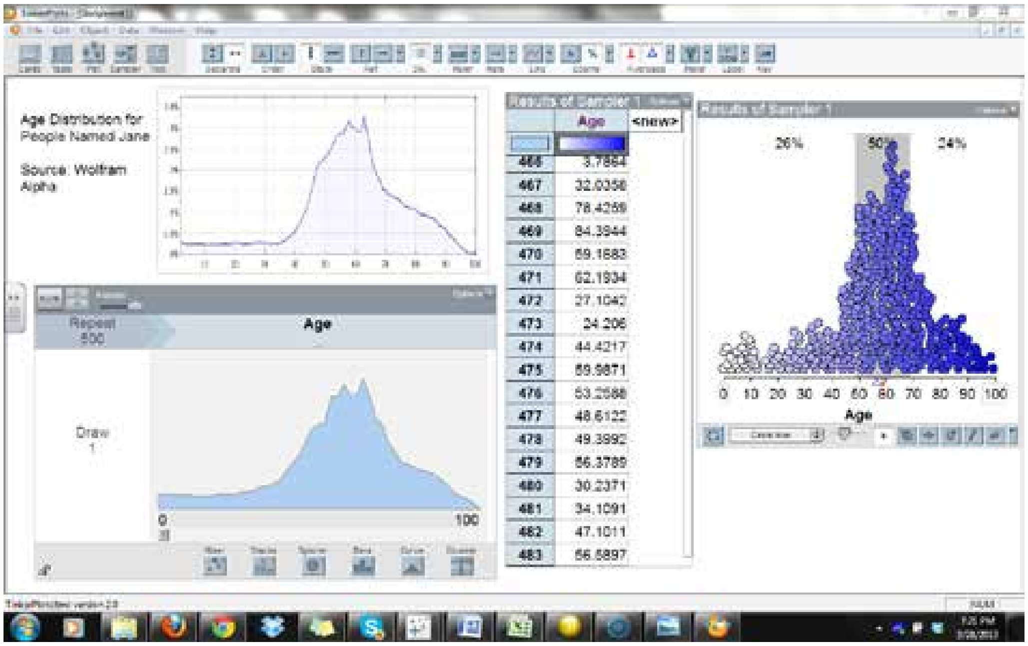

Figure 16. Model for estimating the likely age when meeting a person named Jane.

- How do modeling activities increase students' understanding of how science and mathematics are connected?
- How do students make sense of the congruence or incongruence across different models of the same phenomena? How do they determine which models are most useful?
- How do students conceive of the reasoning they do between data and models as connected between what they experience in the behavior of natural systems?

Interacting with Quantities with Gestures and Multi-touch Environments

Gestures provide a window into a person's conception. If you were asked to describe a "blob", what would you say? Most would not only use a verbal description, but would likely use hand gestures to illustrate "blobbiness." Thus, if we are creating digital experiences that take advantage of gestures through touch and multi-touch capabilities, as well as physical bodily movements, designers need to think carefully about how well the gestures used in an interface map onto the intended conception. Thus, for a mathematical environment, we need to consider the mathematical fidelity of the representations and interactions with those representations (Zbeik, Heid, Blume, & Dick, 2007). Since the interactions include gestures, designers and researchers should attend to gestural congruency as mapping between a gesture (physical embodiment) and the digital representation of the domain (Segal, 2011).

If students are using touch to generate objects on an iPad within certain applications (e.g., Doodle Buddy), they may develop an understanding of one-to-one correspondence (one tap, one object). The use of one-touch actions per object are similar to software that has been used in mathematics where a single mouse click or a drag and drop action is used to place an object representing some quantity into a workspace (e.g., Steffe & Olive, 2002). However, in a multi-touch environment, a user can also create multiple objects by holding down one finger, or using several fingers (or their palm) to generate many objects. For example, Figure 17 illustrates two children simultaneously trying to cover the screen with ladybugs, each using single taps, and developing different strategies for producing many objects rather than one at a time, perhaps to be efficient (Figure 17). I wonder how this embodied action represents their conception of magnitude of quantity?

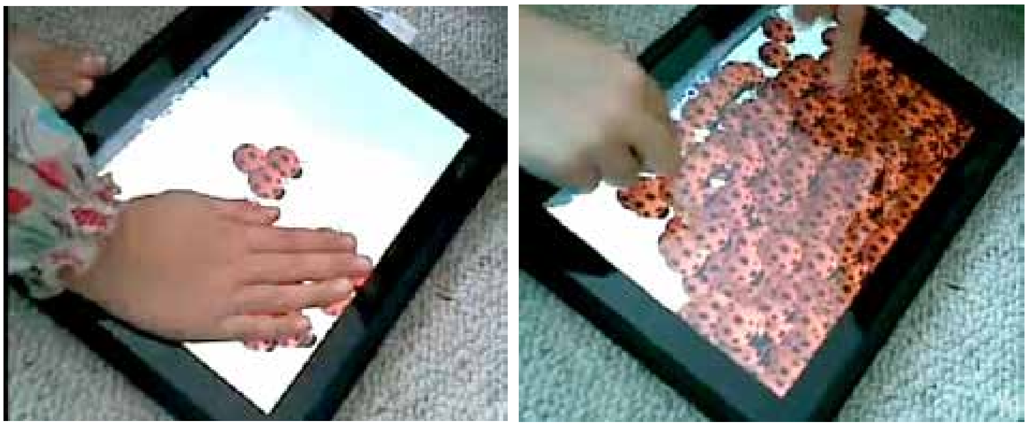

Figure 17. Students transitioning from using a single finger for creating single objects to using a palm to generate many objects.

Other algebraic multi-touch software (e.g., Mathination, Algebra Touch) has also been developed for allowing users to interact with symbolic expressions to factor expressions (two finger pinch), multiply two polynomials (two finger spread), or use a single slide/swipe to move a monomial to a different location in an expression or equation, including moving the object across the equality sign with the appropriate mathematical operation performed automatically. While I was able to find a study that showed improvement in students' ability to solve algebraic tasks (e.g., combining like terms) without structural errors after engagement with the Algebra Touch software (Ottmar, Landy, & Goldstone, 2012), it is not clear how the gestures used in the software are impacting students' understanding of algebraic structure and manipulation. This seems like a needed area of research if such applications are developed and used in schools.

Dynamic geometry environments have been available for two decades and have been widely adopted. Empirical studies show advantages in the use of such software in developing geometric conceptions (e.g., Hollebrands, 2007; Jiang, White, & Rosenwasser, 2011). New dynamic geometry environments are being developed that take advantage of the multi-touch and gesture-based interface of tablets. For example, Sketchpad Explorer allows for using files created using the computer-based Geometer's Sketchpad software, but does not include capabilities of construction and measurement. However, Sketchpad Explorer does allow for multi-touch interactions such as dragging two points simultaneously. Other dynamic geometric construction and measurement software is available for tablet software (e.g., Geometry Pad, Isosceles, Geo Designer). In mid 2012, a group of researchers and developers in Germany released a beta version of a free dynamic geometry environment that uses gestures as its primary user input (www.sketchometry.com). The software runs through a browser and works on PCs, MACs, iOS and Android. Gestures can be initiated with a mouse on a computer or fingers on a touch screen (tablet or interactive whiteboard). Figure 18 illustrates the gestures used to construct a circle (top images), then to construct a line parallel to a given line through a given point (bottom images).

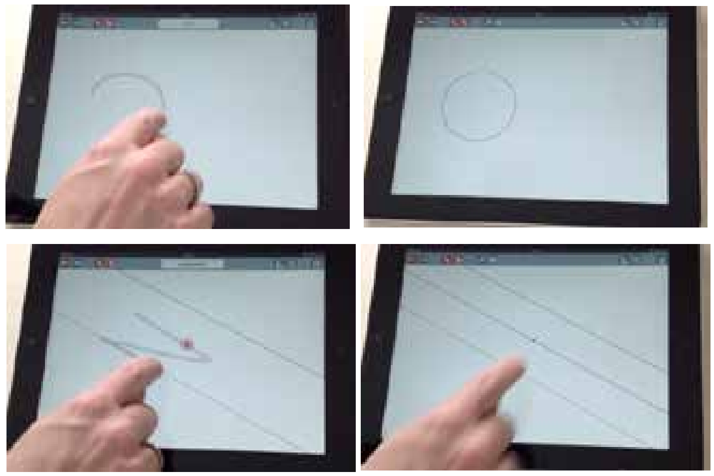

Figure 18. Gestures used in Sketchometry to construct a circle and parallel line through a point.

Physical approaches to conceptions of distance, time, rate of change, and functions have been used in mathematics and science instruction for two decades using special data probeware and motion detector devices. Modern game companies have advanced the use of motion detector technologies for use in game systems such as the Wii and Xbox 360 Kinect system. A team from University of Washington Bothell, led by Robin Angotti, has developed an application called Kinect Math that allows for gestural based interactions with function concepts (kinectmath.org, 2012). The Tracking mode app uses the Kinect in a similar way to motion detector probeware that has been shown to have positive impacts on students' understanding of rate of change (e.g., Lapp & Cyrus, 2000; Radford, Demers, Guzman, & Cerulli, 2003). However, the editing mode within the software allows a user to perform particular gestures to change the parameters of a function. For example, with a quadratic function, moving your right hand horizontally away from the body makes the parabola "fatter", while moving your right hand across your body makes a parabola "skinnier" (Figure 19). These types of gestures are significantly different than the gesture used to swipe or move a slider to control a parameter value like in many software applications.

Figure 19. Student using right hand to perform a transformation with a quadratic function (image from kinectmath.org).

Using gestures through multi-touch interfaces and bodily motions has the potential to develop new practices for understanding of quantities and how they are represented in the world. While gesturing is not a new practice for humans, students with many experiences using gestures to control objects may develop new understandings that are connected to these physical experiences.

- What conceptions do students develop when using gesture based applications to interact with representations of quantitative information? Do these conceptions differ from students' conceptions when using different technologies? For example,
  - » How do students' conceptions of circles and parallel lines differ when using gesture based software such as Sketchometry versus using a mouse to select objects and menu-driven construction processes with software such as Cabri, Geometer's Sketchpad, or GeoGebra?
  - » How do students understand function transformations differently when using free-form gestural interfaces (a Kinect) to interact directly with a graphical representation as opposed to dragging or swiping to change the value of a parameter using a slider?
- How can adaptive technologies be developed that allow students with physical challenges to interact in similar ways?
- How can virtual reality environments change the ways that students and teachers interact with each other through gestures?

Towards the Future

Rapidly changing technologies will demand that we are continually considering how new practices and knowledge may evolve within the domain of quantitative reasoning. These are exciting times with the potential for many opportunities for researchers, teachers, designers, and digital developers to interact in meaningful ways to push the field forward. Research with these new digital tools and students' learning and their emergent practices should inform the development of new learning theories and proposed learning progressions that can and should inform the future standards and curriculum efforts in mathematics and science education. I hope that some of the examples described in this paper have peaked your curiosity and provided a few pebbles, or stepping stones, that can take us on new research frontiers.

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