COHERENCE, QUANTITATIVE REASONING, AND THE TRIGONOMETRY OF STUDENTS

Kevin C. Moore
University of Georgia

Abstract

Over the past five years I have sought to better understand student thinking and learning in the context of topics central to trigonometry, including angle measure, the unit circle, trigonometric functions, periodicity, and the polar coordinate system. While each study has provided unique insights into students' learning of trigonometry, a common theme connects the studies' findings: quantitative reasoning plays a central role in students' trigonometric understandings. In this chapter, I first describe a coherent system of understandings for trigonometry that is grounded in quantitative reasoning. Against this backdrop, I compare students' quantitative reasoning in the context of trigonometry in order to illustrate the role of quantitative reasoning in the learning of a particular mathematical topic.

Coherence, Quantitative Reasoning, and the Trigonometry of Students

Whether focused on classroom discourse, student learning, or teacher knowledge, mathematics education research efforts have long shared the common goal of determining how to engender coherent mathematical experiences for students. Despite holding this shared goal, it is not clear that there is a shared image of what *coherence* means in the context of mathematical ideas and student learning. As Thompson described, "One would think that with the increasing emphasis on curricular coherence, everyone would be clear on how to think about it. This is, unfortunately, not the case" (2008, p. 47).

Trigonometry is an example of a mathematics topic that lacks coherence in mathematics education. Students are expected to understand and flexibly use trigonometric functions in multiple contexts, including right triangles and the unit circle. Yet, students and teachers often do not construct meanings for trigonometric functions that include robust connections between these two contexts (e.g., Akkoc, 2008; Thompson, Carlson, & Silverman, 2007). Students' and teachers' difficulties in trigonometry suggest that current curricula approaches to trigonometry do not engender coherent student understandings.

In an attempt to address the apparent shortcomings in the teaching and learning of trigonometry, I conducted a series of studies that aimed to identify ways of reasoning supportive of coherent trigonometric understandings. These studies have ranged from exploring how precalculus students' angle measure understandings influence their learning of trigonometric functions (Moore, 2010, submitted) to investigating pre-service teachers' notions of the unit circle (Moore, LaForest, & Kim, in press) and the polar coordinate system. While each study has offered different insights into students' trigonometric thinking, a common theme emerged in each case: quantitative reasoning (Smith III & Thompson, 2008; Thompson, 1989, 2011) was critical in supporting the participants' construction of a coherent system of trigonometric ideas.

The present chapter illustrates the central role of quantitative reasoning in trigonometry. I hope to accomplish a few goals, each of which stems from Thompson's (2008) description of conceptual analysis, by discussing quantitative reasoning in the context of trigonometry. First, I describe a trigonometry that is built on a coherent system of meanings grounded in quantitative reasoning. In doing so, I provide an example of how theories of quantitative reasoning can act as a tool for characterizing a coherent system of meanings for a mathematical topic. The second goal of this work is to exemplify several central tenets of quantitative reasoning by illustrating these tenets in the context of the trigonometry of students.

Describing a Coherent Trigonometry

Any attempt at describing coherence must include more than a progressive list of topics and definitions. If focused solely on a list of topics, an educator can fall into the trap of treating mathematics as composed of segmented and mostly independent goals of learning. Identifying a sequence of topics is no doubt a component of developing coherent mathematical experiences for students, but coherence results from, "…the development of meanings of each [topic] and the construction of contextual inter-relationships among them" (Thompson, 2008, p. 47). As Thompson (2008) highlighted, emphasis must be placed on meanings and ideas, the compatibility of these meanings, and how certain meanings support (or inhibit) the construction of subsequent meanings. The introduction of angle measure before trigonometric functions certainly makes sense as a progressive sequence of topics, but this sequencing of topics alone is not equivalent to coherence. Instead, coherence is the product of angle measure meanings that support students' construction of connected understandings of trigonometric functions.

Some Textbook Definitions of Angle Measure and Trigonometric Functions

Below I present several definitions of angle measure (Table 1) and the sine function (Table 2) that reflect many of those commonly printed in US textbooks. As can be inferred, there are numerous definitions for each topic, with each definition conveying different imagery. Definition 1 (Table 1) approaches angle measure as the result of using a protractor, while Definition 2 (Table 1) describes angle measure in terms of the physical act of a rotation. Relative to the sine function, Definitions 1 and 2 (Table 2) present the sine function as a coordinate, while Definitions 3 and 4 (Table 2) present the sine function in terms of a ratio. The definitions presented in Tables 1 and 2 also mention several trigonometry contexts, with each context having its own purpose and place in mathematics and related fields.

Definitions of Angle Measure	
1	An angle is formed by two rays with a common endpoint. You can measure an angle in degrees with a protractor (*picture of protractor included with definition*).
2	A measure of one degree is equivalent to a rotation of 1/360 of a complete revolution.
3	An angle of 1 degree is 1/360 of a circle.
4	Radians are a unit of measurement for angles such that 2π radians correspond to a rotation through an entire circle.
5	The radian measure of an angle is the ratio of the arc cut off by the angle to the radius of any circle centered at the vertex of the angle.
6	An angle of 1 radian is defined to be the angle at the center of a unit circle that spans an arc length of 1.
7	The radian measure of an angle is the distance traveled along the unit circle from the initial side of the angle to the terminal side of the angle.

Table 1. Definitions of angle measure

Definitions of the Sine Function	
1	The sine of an angle θ, denoted sin θ, is defined to be the second coordinate of the endpoint of the radius of the unit circle that makes an angle θ with the positive horizontal axis.
2	Consider a circle with a radius 1 centered at the origin. If $P=(x, y)$ is a point on the unit circle that forms an angle θ with the origin and the point (1,0), then $\sin \theta = y$.
3	For any right triangle that has an acute θ angle , the sine of θ , abbreviated sin θ, is the ratio of the length of the leg opposite this angle to the length of the hypotenuse.
4	Let θ be an acute angle of a right triangle. Then $\theta = \frac{\text{opp}}{\text{hyp}}$.

Table 2. Definitions of the sine function

My purpose for providing these definitions is to illustrate the complexities involved when attempting to concisely define trigonometric functions and angle measure (or any mathematics topic for that matter). Definitions intend to capture a meaning for a topic or idea. But, cognitively, meanings are composed of a complex system of schemes and operations. For instance, students' meanings for ratio (Definitions 3 and 4, Table 2) are wide ranging (Thompson & Saldanha, 2003), and these differing meanings have significant implications for how students' interpret such a definition. As Tall and Vinner (1981) described, a (concept) definition is no more than a sequence of words and symbols. What is most important is the cognitive structure that is evoked by a definition, and this cognitive structure is the product of years of experience.

The fact that the above definitions reference several mathematical topics and contexts emphasizes the connectedness of mathematics. Coherence must attend to the fit of several topics' meanings, which can't be accomplished by a list of topics or a sequence of textbook definitions. When describing a coherent trigonometry in the following sections, I avoid an emphasis on definitions that are in the spirit of those listed above. Instead, I aim to describe a system of meanings that enable reasoning about trigonometric functions, regardless of the context, in compatible ways (at least at the introductory level).

It Starts with Angle Measure

Students' and teachers' difficulties in trigonometry often stem from impoverished angle measure understandings (Akkoc, 2008; Moore, 2010, submitted; Topçu, Kertil, Akkoç, Yilmaz, & Önder, 2006). Providing a possible explanation for this phenomenon, Bressoud (2010) and Thompson (2008) identified that US curricula commonly treat angle measure and trigonometric functions in ways that create incompatible meanings for degree and radian angle measures. These incompatible meanings form an inherent divide between trigonometry contexts, and this divide is reinforced by the predominant use of degree measures with right triangle trigonometry and radian angle measures with unit circle trigonometry.

A student's first experience with angle measure tends to begin with the *use* of a protractor to determine an angle measure (Definition 1, Table 1). Students are taught to use a protractor, classify several types of angles (e.g., acute, supplementary, complementary, and vertical), and perform calculational strategies to relate the measures of these types of angles. Such calculational strategies and the relationships driving these strategies are important for geometry, but at the same time this calculational approach to angle measure does not address the structure of the unit used to measure an angle's openness. Teaching students to use a protractor without addressing the meaning of the measure in relation to the structure of the measuring unit circumvents the *quantification* (Thompson, 2011) of angle measure. A likely consequence of pedagogical approaches that fail to properly address the quantification of angle measure is that students fail to develop a discernable concept of angle measure (Moore, submitted).

Contrary to the teaching of degree angle measure, common treatments of radian angle measure do not include measuring angles with a protractor. In fact, one is hard pressed to find a protractor for measuring angles in radians.[1] Instead, radian angle measure is defined in terms of a multiplicative relationship between the arc length subtended by an angle's rays and the radius of the corresponding circle, which is represented by the formula $s=r\theta$. Radian measure is also introduced well after a student's first experience with degree measure, likely leading to the situation where students spend years developing notions of degree angle measure that differ significantly from intended radian measure meanings. In such a situation, it should come as little surprise that students and teachers develop disconnected and shallow angle measure understandings that are dominated by degree angle measure (Akkoc, 2008; Topçu et al., 2006).

If degree and radian angle measures are measures of the same thing – the openness of an angle – and we expect students to develop compatible meanings for these units, then instruction should approach both units in the same manner from the outset. As Bressoud (2010) and Thompson (2008) described, arc length ideas for angle measure offer a way to address both units in the same manner.[2] Angle measures, in either unit, can be thought of as conveying the fractional amount of any circle's circumference subtended by the angle's rays, provided that the circle is centered at the vertex of the angle. Any unit that is proportional to the circle's circumference can be used to measure an angle, with standard units stemming from convenience (e.g., radians and unitizing a circle's radius length) or contextual characteristics (e.g., degrees and planetary motion). An angle that measures one degree subtends 1/360 of the circumference of any circle centered at the vertex of the angle and an angle that measures one radian subtends $1/2\pi$ of the circumference of any circle centered at the vertex of the angle.

Developing angle measure as the fractional amount of any circle's circumference subtended by an angle's rays enables defining both units of angle measure in terms of the same quantitative relationship. Equivalent degree and radian angle measures convey the same fractional amount of a circle's circumference that is subtended by the angle's rays (if d degrees is equivalent to r radians, then $\frac{d}{360}=\frac{r}{2\pi}$). As with any pair of units that measure the same quantity, degree and radian units are just scaled versions of each other (Thompson, 2008) and should be treated as such if students are to connect measures in both units.

More on Measuring in Radii

Textbooks often define trigonometric functions in the context of the unit circle, which is presented as a circle with or a radius of one (Definitions 1 and 2, Table 2). Little to no attention is given to what "1" represents, how "1" might be related to a circle's radius that is not equal to "1" (e.g., a circle with a radius of 57 feet), or how "1" relates to measuring the circumference of a

1 At the time of writing this chapter, I could not find a protractor for purchase on Google (www.google.com/shopping) or Amazon (www.amazon.com) that measures in radians.

2 I have been asked why both units can't be defined in terms of an amount of rotation. Rotation imagery can certainly be used for the openness of an angle, but such imagery still leaves the question of, "What do we mean by a rotation of a degrees, and how do we determine when a ray has rotated a degrees from another ray?" These questions raise issues of how one quantifies a rotation, issues that can be addressed through measuring arc lengths. I also note that in working with students, I've found that rotation imagery is a natural byproduct of a quantification process that involves measuring along arcs.

circle in radii. I expect that this leads to students conceiving of "1" as a *number* (e.g., a number that does not reflect the result of a measurement process), as opposed to a *value* (e.g., a number that represents the result of a measurement process).

A specified value represents a measure and necessitates a unit associated with the value (Thompson, 2011). An approach that develops "1" as a value is to consider the unit circle as the product of using the radius of a circle as a unit of measure. If given a circle with a radius length of measure *c* when measured in a unit of magnitude $||u||$, then using the length of the radius as a unit of measure amounts to measuring in a unit of magnitude . Measuring in a unit of magnitude $c||u||$ yields measures that are $\frac{1}{c}$ times as large as equivalent measures in a unit of magnitude . Thus, when measured in a unit of magnitude $c||u||$, the radius has a measure of $\frac{c}{c}$, or 1 (Figure 1).[3] In this case, the "1" represents a measure in radii, and the given circle can now be thought of as a unit circle: a circle with a radius of length 1 radii. The circle's circumference can also be measured in radii, yielding a measure of 2π radii; the process of measuring in radii connects the unit circle to radian angle measures. From here, the radius can be thought of as the unit of measure for distances from the center of the circle, which produces the coordinate pairs associated with the unit circle.

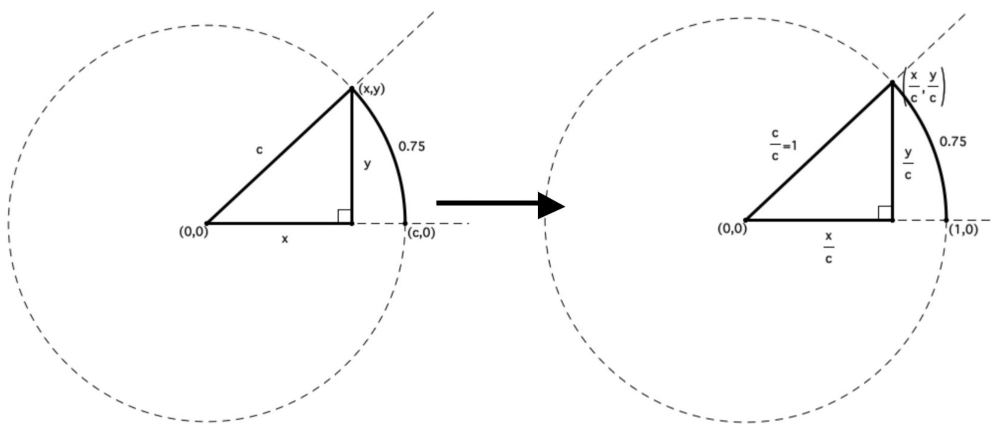

Figure 1. Using the radius as a unit of measure

Developing a meaning for the unit circle that is rooted in measuring lengths in radii also enables defining the outputs of the sine and cosine functions (e.g., coordinates on the unit circle) as measures in radii. The outputs are *values*; they represent measures that entail a unit. $\sin(0.75) \approx 0.682$ means that for a point that is an arc length of 0.75 radii counter-clockwise from the 3 o'clock position on a circle, the vertical distance above the circle's horizontal diameter is approximately 0.682 radii (0.682 times as large as that circle's radius). The treatment of the output of the sine and cosine functions as measures in radii also provides a natural avenue for addressing the amplitude of the sine and cosine functions. To determine the coordinate pair in Figure 1, we first determine the appropriate vertical and horizontal distances in radii (e.g., (cos(0.75),

3 This measure-magnitude approach to unit conversion is described by Thompson (2011) and stems from Wildi's discussion of magnitude (1991). Instead of placing an emphasis on dimensional analysis (e.g., unit-cancellation), focus is shifted to coordinating how the measure of a quantity changes as the magnitude of the unit used to make the measure changes.

sin(0.75))) and then multiply these measures by the length of the radius measured in the desired unit (e.g., $r\cos(0.75)$, $r\sin(0.75)$. Generalizing this approach for an angle measure θ, we obtain (x,y) = ($r\cos(\theta)$, ($r\sin(\theta)$)). In this case, the amplitude of the sine and cosine functions represents a conversion to measures in whatever unit one chooses to measure the radius.

Connecting Trigonometry Contexts

At this point, I haven't addressed trigonometric functions in a right triangle context, instead focusing on a circle context to discuss angle measure and measuring in radii. The trigonometric ratios for the output of the sine and cosine functions naturally surface when the unit circle is developed through a process of measuring in radii (Figure 1), but the question of how a circle-less right triangle (Figure 2) is connected to the unit circle remains.

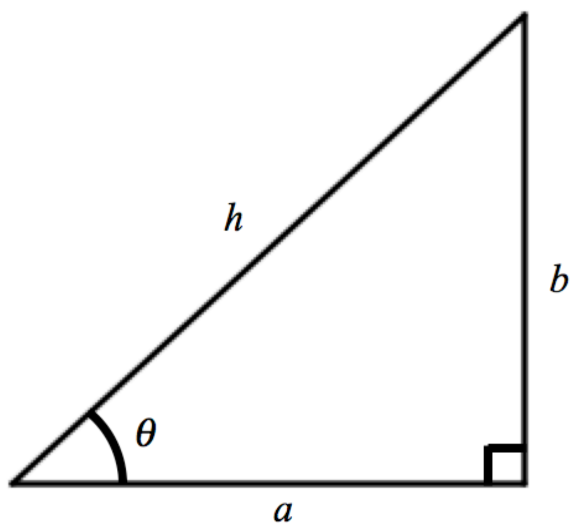

Figure 2. A standard right triangle trigonometry diagram

The aforementioned approach to angle measure necessitates constructing a circle centered at the vertex of the angle to be measured. Thus, when presented with the angle measure θ in Figure 2, I intend that the reader imagine a family of circles (or at least one circle) centered at the vertex of the angle. As one option, the arc used to denote the angle of measure θ can be extended to form a circle centered at the vertex of the angle. As another option, the hypotenuse of the right triangle can be chosen as the radius of a circle (Figure 3). This second option leads to the situation presented in Figure 1, and the trigonometric ratios emerge from using the hypotenuse (a radius) as a unit of measure for the legs of the right triangle. For example, the output of the sine function can be thought of as a measure in hypotenuse (or radius) units of the leg opposite to the angle (e.g., $\sin(\theta) = \frac{b}{h}$ or $h\sin(\theta) = b$, θ in radians). At this point, ideas of similarity can be used to consider the implications of increasing or decreasing the length of the hypotenuse (e.g., the leg lengths, measured in hypotenuse units, are constant for a constant angle measure).

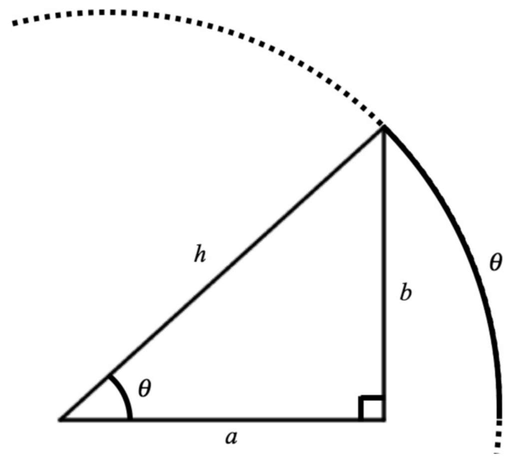

Figure 3. Using the hypotenuse as the radius of a circle.

By leveraging various measurement ideas and arc meanings for angle measure, right triangle trigonometry can build on the aforementioned ideas of unit circle trigonometry. The sine and cosine functions relate angle measures (with radians being the standardized unit) and lengths that are measured in a particular unit (the hypotenuse or radius). Additionally, directly tying angle measures to arcs so that students imagine a circle along with an angle measure offers a natural connection between the two settings, with the hypotenuse of a right triangle being one choice for the radius of a circle.

Before continuing, I note that this approach to trigonometry heavily relies on measurement, which illustrates that coherence involves the fit of meanings. Students' measurement schemes are non-trivial and develop over years of experience (Steffe, 1991a, 1991b; Steffe & Olive, 2010). A student's notions of measurement will critically influence her ability to construct meanings compatible with those previously described, and coherence would be the result of instruction developing measurement notions at an early age that support the aforementioned approach to trigonometry.

Quantitative Reasoning and the Trigonometry of Students

The previous discussion outlines a few central trigonometry ideas, but it does not address how the described ideas might develop or emerge when working with students. Working with students is necessary to uncover unforeseen ways of thinking and characterize the implications of these ways of thinking. As Steffe and Thompson (2000) described, working with students places a researcher in a situation where he can experience constraints in the students' ways of operating. These experiences enable the researcher to modify his notion of what meanings and ways of reasoning are central to the development of a coherent understanding for a topic, while also determining how various meanings might develop.

What follows is a discussion that highlights student reasoning in the context of the aforementioned ideas of trigonometry. Specifically, I present data from a series of studies (Moore, 2009, 2010, submitted; Moore et al., in press) that illustrate how these ideas play a critical role in students' learning of trigonometry. In addition to characterizing instances in which students appeared to reason in ways compatible with the aforementioned ideas, I discuss occasions during which students did not act in ways compatible with these ideas. Collectively, the instances of students' actions demonstrate the centrality of quantitative reasoning relative to foundational trigonometry concepts.

Angle Measure: Quantitative Relationships vs. Labels

Thompson (2008, 2011) claimed that common curricula approaches to angle measure do not promote clear and coherent meanings for angle measure. Particularly, these approaches do not support the quantification of angle measure in a discernable way. Corroborating Thompson's characterization of curricula approaches to angle measure, I identified that precalculus students hold multiple disconnected meanings for angle measures, where these meanings often depend on the measure under consideration (Moore, 2009, 2010, submitted). When given angle measures like 90 degrees, 180 degrees, or 360 degrees, the students referenced perpendicular lines, straight lines, and circles, respectively; angle measures were inherent properties of geometric objects, as opposed to specified values resulting from a measurement process. For angle measures (e.g., 1 degree or 43 degrees) that did not invoke geometric objects, the students regularly described calculations to determine supplementary and complementary angles (e.g., angles that formed these geometric objects with an angle of the given measure); angle measures were defined by calculational relationships with other angle measures.

The students' lack of a discernable angle measure concept prevented them from solving various tasks that required them to reason about an angle measure independent of other angle measures, such as determining ways to measure an angle without a protractor. In response, I worked with the students to develop the angle measure meanings outlined in the previous section. A finding of the study was that the students' quantification of angle measure was gradual and required that they frequently reflect on their actions, particularly to give meaning to executed calculations and numbers. When solving the instructional activities, it was common for the students to determine specified values and execute calculations involving these values (e.g., dividing an arc length by a circle's radius or circumference). But, the students infrequently reflected on their calculations in terms of a structure of related quantities. Instead, the students typically sought to determine subsequent calculations, giving little consideration to determining a meaning fore the executed calculation. This approach to solving the problems often resulted in the students failing to give meaning to numbers or keep track of a meaning for a determined number. Thus, it became necessary that I prompt the students to reflect on and consider the meaning of their calculations independent of the numerical result of the calculation.

As an example, when creating protractors of different sizes during an instructional activity, the students determined the circumferences of the protractors and the corresponding arc lengths associated with one unit of angle measure. At this time, I prompted the students to determine a meaning for dividing subtended arc lengths by the circumference of the corresponding circle. After noting that the ratios were equivalent, the students concluded that the ratios conveyed that the angle subtended an equivalent fractional amount of each circle's circumference. This realization supported the students in concluding that an angle measure conveys the fractional amount of *any* circle's circumference subtended by the angle. Opportunities like this were instrumental in supporting the students' quantification of angle measure. Specifically, the students' reflection on their calculations in terms of the quantities of the situation created occasions during which they could abstract meanings for angle measures involving quantitative relationships between unspecified values.

To illustrate student actions that came as a consequence of constructing angle measure meanings that were comprised of quantitative relationships, consider Zac's solution to the "Arc Length Problem" (Figure 4) (Excerpt 1).

Excerpt 1

Zac:Um, ok. So what I plan on doing for this one is converting thirty-five degrees into radians. And a very easy way of doing that is putting thirty-five over three sixty is equal to *x* over two pi (*writing corresponding equation*)…And then with that all I have to do is just multiply the answer (*pointing to x*) by two inches, two point four inches, and two point nine inches (*pointing to each value in the problem statement*) to get the different arc lengths (*identifying each arc length with his pen tip*) right there, because radians is just a percentage of a radius. (*Zac executes described calculations and interviewer asks him to explain his angle conversion formula*). Well what you're doing is just technically finding a percentage. Like thirty-five over three sixty is (*using calculator*), is nine point seven percent of the full circumference.

Zac's actions suggest that his angle measure meanings entailed quantitative relationships, and these relationships enabled Zac to play out his solution without executing calculations. Specifically, Zac first reasons that angle measures convey a fractional amount of circle's circumference to convert between two angle measures (lines 1-3 & 9-11). Then, without determining a specified radian angle measure, he reasons that radian measures convey a multiplicative relationship between the arc lengths subtended by the angle and the corresponding radius to *anticipate* determining the undetermined arc lengths.

In contrast to reasoning about angle measures as values that entail quantitative relationships, another student (Amy) involved in the same study as Zac predominantly reasoned about angle measures as labels of geometric objects. Amy referenced angle measures as "part of a circle," yet her actions did not suggest reasoning about arc lengths as measurable and multipli-

Given that the following angle measurement θ is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches (*figure not to scale*).

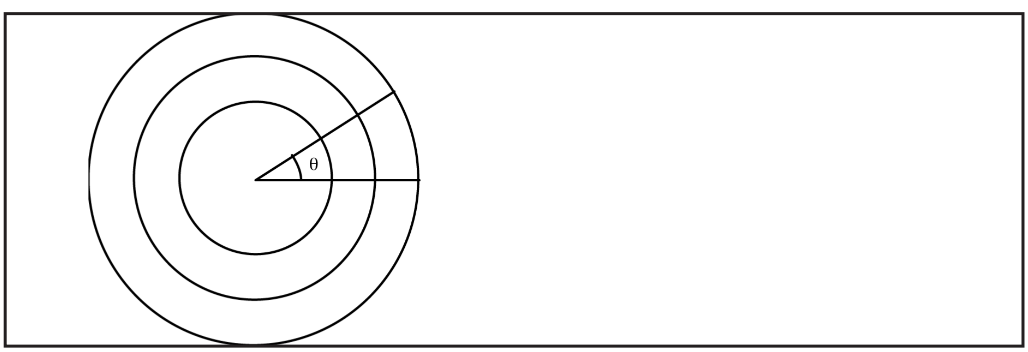

Figure 4. Arc Length Problem

catively comparable to other lengths (e.g., the circumference). Amy's actions also implied that, in the context of angle measure, she had not made a discernable distinction between areas and arc lengths as measurable attributes. Amy's angle measure notions had significant implications for her learning of trigonometric functions. As an example, consider her description of π when interpreting the equation cos(π) = -1 (Excerpt 2).

Excerpt 2

Amy: Ok, well, we have cosine pi and you get negative one. So usually when you have, I mean what we've done in the past, like up here, is the positive side (*tracing the top half of a circle*). And down here is like negative (*pointing to the bottom half of a circle*). Like if we do it on like a literal gra-, literal graph. It will, I'm going to far into it, never mind. Um, this is like plus one and this will be like negative one (*writing each value by the corresponding part of the circle*). And then since pi is half a circle (*tracing the bottom half of a circle*), when I see cosine pi, to me, that means, like the bottom half of the circle, is what it represents.
KM: So what do we mean by...
Amy: So negative one, like radius, at the bottom half since...
KM: So the pi because it's negative now represents the bottom half.
Amy: Ya, of the circle.

During this interaction, Amy's interpretation of numbers as labels or referents of geometric objects (as opposed to values that entail multiplicative relationships) is prevalent as she describes π as "half a circle," with the number -1 signifying the "bottom half" of a circle. After this interaction, Amy added, "Ok, cosine of pi is just half the circle." Such reasoning inhibited Amy's ability to interpret statements like sin (2) ≈ 0.91. Another implication of Amy's reasoning was that it did not support conceptualizing graphs as a collection of paired values that represent how these values vary in tandem. Instead, she interpreted graphs of the sine and cosine functions as representing the "top half" and "bottom half" of circles. On the other hand, Zac conceived of the input of the sine and cosine functions as the measure of an arc in radii, which subsequently supported him in coming to understand the sine function as representative of how two quantities' values vary in tandem (described below).

Unit Circle: A Circle of Radius One vs. a Circle of One Radius Length

The approach to the unit circle outlined earlier requires coordinating changes in a quantity's measure with changes in the magnitude of the unit used to measure the quantity. This approach to unit conversion stands in stark contrast to more traditional approaches to unit conversion. For instance, dimensional analysis (or "unit-cancellation") is commonplace in mathematics, physics, and engineering education. Dimensional analysis typically proceeds as follows:

1)I have a measure, 3 meters, that I need to convert to another unit of measure, a number of feet.

2)There are approximately 3.28 feet in one meter (or there are approximately 0.305 meters in one foot).

3)I use the calculation of $3\ \cancel{\text{meters}} \bullet \frac{3.28\text{ feet}}{1\ \cancel{\text{meter}}} \approx 9.84\text{ feet}$

(or) $8\ \cancel{\text{meters}} \bullet \frac{1\text{ feet}}{0.30\ \cancel{\text{meters}}} \approx 9.84\text{ feet}$ because the units cancel appropriately.

This approach to unit conversion circumvents coordinating relationships between the magnitude used to measure a quantity and the measure.[4] As Thompson (1994a) described, dimensional analysis is not much more than a "formalistic substitute for comprehension" that is designed to "get more answers" (p. 226). Confirming Thompson's claim, Reed (2006) identified that (despite expecting the contrary) dimensional analysis can mask important mathematical ideas and lead to decreases in student performance.

When working with a pair of students (Bob and Mindy) enrolled in an undergraduate secondary mathematics content for teaching course, the importance of measure-magnitude reasoning and the lack of meaning behind dimensional analysis came to a head. This occurred as the students attempted to use the unit circle to solve angle measure and trigonometry problems (Moore et al., in press). Specifically, both Bob and Mindy considered the unit circle as distinct from given circles and their attempts to use dimensional analysis often inhibited their ability to relate given circles to the unit circle.

During the first session with each student, we presented circles with given radius lengths and subtended arc lengths (Figure 5) and asked the students to determine the corresponding angle measures. *Both* students[5] first drew a second circle with "a radius of one" and referred to the given circle as the "original circle" and their drawn circle as "the unit circle."

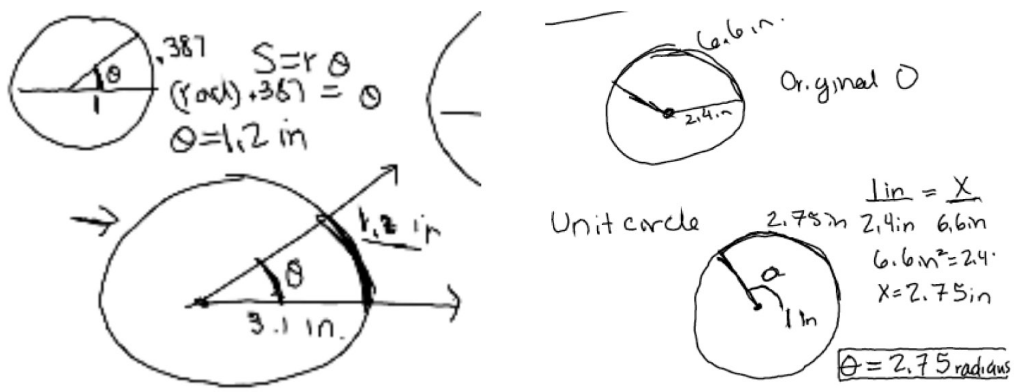

Figure 5. Bob' (left) and Mindy's (right) initial use of the unit circle

As Mindy solved the problem, she claimed, "So this is our original circle and this is going to be a unit circle. We know that by nature, a unit circle is going to have a radius one. Because we are already given the unit, we can go ahead and say one inch." Mindy then set up an equation (Figure 5) to solve for the undetermined arc length, while comparing the units of each value to conclude that she had set up the equation correctly (e.g., the units cancelled in the initial equation so that x represented a number of inches). After determining x in inches, she changed the units of her answer to radians, claiming that the problem asked for an angle measure. Like Mindy, Bob attempted to determine a unit associated with the unit circle. Differing from Mindy, Bob first divided the given measures by the given radius length. As a result of dividing measures

4 As an aside, when compared to the metric system, US customary units appear quite arbitrary and I expect that students are often burdened with the task of attempting to remember things like how many yards are in a mile, as opposed to focusing on coordinating magnitudes and measures. With the metric system, each unit magnitude is always some power-of-10 times as large as any other unit magnitude. This feature of the metric system might enable an increased instructional focus on coordinating magnitudes and measures, as there is a decreased need to memorize the relationship between magnitudes.

5 We met with each student individually.

of like units (e.g., the given radius value by the given radius values to obtain "one" for the radius), he encountered difficulties using dimensional analysis to track and justify his calculations. Bob became particularly confused as he attempted to use the formula s = rθ and relate units for the variables of *s*, *r*, and θ for both circles.

Bob and Mindy's actions suggest that their notions of the unit circle were not inherently tied to using the radius as a unit of measure. The unit circle, to them, was a circle of "radius one," but it was not a circle with a radius of *one radius length*, where this length formed the unit of measure for quantities other than the radius. Thus, it is likely that the students did not think to use the *given* circle's radius as a unit of measure, and instead they chose to draw a circle with a "radius of one." Although Bob properly divided by the radius, he did not conceive of this operation as representative of measuring quantities in radii. Mindy arrived at the correct solution, but was unable to explain why it was appropriate to change the units to her answer. When I later asked Mindy to explain her method of cancelling the values' units, she claimed, "It's really about comfort because I don't like to do something unless I can see the units perfectly dividing out."

An implication of divorcing the unit circle from given circles was that the students encountered difficulty relating trigonometric functions to circles with a radius length not equal to "one." For instance, the students could identify that sin(0.5) ≈ 0.48 implies that the *y*-coordinate on the unit circle is 0.48 at an angle measure of 0.5 radians (Definition 2, Table 2). However, the number 0.48 did not entail a unit of measure, and thus this number did not lend itself to the method of dimensional analysis.

In response to (a) the students' propensity to reason about the unit circle as distinct from given circles, (b) their focus on dimensional analysis when performing calculations, and (c) their absence of associating the unit circle to measuring in radii, we engaged the students in instructional activities designed to develop unit conversions through the magnitude-measure approach previously described. We intended that the students come to understand the unit circle as the result of choosing the radius as a unit of measure, which we conjectured would necessitate that the students view lengths (e.g., the radius) as taking on multiple measures all at once (e.g., measures in radii and measures in feet).

As an example task, we gave the students a picture of a stick and asked questions along the lines of:

1) What does it mean for the stick to have a length of 3.4 feet?
2) Given that there are 12 inches in one foot, how long is the stick when measured in inches? Given that there are 300 feet in a football field, how long is the stick when measured in football field lengths? Given that a *fraggle* is a unit of measure that is 221 times as large as 2 feet, what is the length of the stick when measured in fraggles?
3) When answering the above questions, did the length of the stick change?

We also imposed the restriction that the students could not use a formula, written expressions, or dimensional analysis when answering each question and we designed the fraggles question to counter dimensional analysis. We followed these questions with a similar task that focused on measuring arc lengths and a circle's radius in different unit magnitudes, including a magnitude equal to the radius length.

When solving the stick task, we observed the students making no distinguishable difference between the magnitude of the unit and the measure of a quantity in that unit. This presented them problems when attempting to solve the fraggles question. As they progressed, they reconciled unexpected results by coming to understand a quantity's measure and the magnitude of the unit used to make the measure as inversely related (e.g., if the unit magnitude is doubled, the measure is halved). In the context of a circle, the students concluded that measures in radii are determined by dividing given measures by the measure of the radius to account for the length of the radius becoming the unit magnitude. After the students made this observation and noted that quantities can simultaneously have measures in radii and other units (e.g., question 3 above), we presented the students with the "Which Circle?" problem (Figure 6).

> Consider circles with a radius of 3 feet, 2.1 meters, 1 light-year, 1 football field, and 42 miles. Which, if any, of these circles is a unit circle?

Figure 6. Which Circle?

Mindy provided the following response (Excerpt 3).

Excerpt 3
Mindy: The unit circle doesn't even require a specific unit other than the radius, I guess that's why it's called the unit circle is like the radius is always just one unit… If we made three feet our radius we would just think about the circle in terms of radii instead of feet then it would be a unit circle. Every circle has a radius, so if you just want to talk about the circle in terms of that unit the radius then every circle is a unit circle…as long as you are considering that the radius is one unit (*holding her hands apart to signify a length*). Like perhaps it's not one unit, I mean it's not one meter in length, but it's one radius in length.

In this example, Mindy describes the unit circle as directly tied to using the radius as a unit of measure. She also notes that every circle can be thought of as the unit circle, regardless of the measure of the radius in other units. Mindy's present explanation of the unit circle suggests different reasoning than her previous actions (e.g., drawing a second circle and labeling the radius with a measure of one in a specified unit). In the present case, she understood that each circle's radius could simultaneously be measured in the given unit or in a unit equivalent to the radius length.

Bob gave a compatible explanation to Mindy's, suggesting that both students conceptualized the unit circle as dependent on using the radius as a unit of measure. As the students moved forward in the teaching sessions, we observed the students no longer considering the unit circle as distinct from given circles. As opposed to only relating the sine and cosine functions to a circle of radius "one," the students' conception of the unit circle supported the understanding that both the input and output of the sine and cosine functions are measured in radii, and that these functions are applicable to any given circle. For instance, $\sin(0.5) \approx 0.48$ now implied that a point 0.5 radii from the 3 o'clock position on a circle is approximately 0.48 radii above the center of the circle, and the students used such reasoning to apply trigonometric functions to any given circle.

Creating Graphs: Connecting Points vs. Covarying Quantities

During the earlier outline of central trigonometry concepts, I did not discuss ways of reasoning that might support graphing the sine and cosine functions, or how the presented system of meanings might support conceiving of these functions in ways that support graphing. *Covariational reasoning*, defined as the cognitive activities involved in coordinating how two quantities vary in tandem, is critical for supporting students' ability to create and interpret graphs as representations of the relationship between two quantities' values (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008). Covariational reasoning is also central to students' learning of major precalculus and calculus ideas (Carlson et al., 2002; Castillo-Garsow, 2010; Confrey & Smith, 1995; Oehrtman et al., 2008; Thompson, 1994b).

When working with a group of precalculus students, covariational reasoning emerged as critical to their ability to conceive of and represent the relationship between two quantities' values (Moore, 2010). As previously discussed (Excerpt 2), one student's (Amy) angle measure notions did not support a conception of two covarying quantities. In fact, Amy did not conceive of angle measure in a way that enabled her to sensibly imagine a varying measure, which led her to produce a graph based on the shape of the context. That is, as previously described, Amy created a graph that perceptually matched the given circle.

In contrast to Amy's actions, consider Zac's solution to the "Ferris Wheel Problem" (Figure 7). Zac did not first create a graph. Instead, he drew a diagram of the situation (Figure 8) and then described a covariational relationship between the relevant quantities (Excerpt 4).

Excerpt 4

Zac:Ok. So a really easy way to do this is divide it up into four quadrants (*divides the circle into four quadrants using a vertical and horizontal diameter*). 'Cause were here (*pointing to starting position*), for every unit the total distance goes (*tracing successive equal arc lengths*), the vertical distance is increasing at an increasing rate (*writing i.i.*)…Then, uh, once she hits thirty-six feet, halfway up, it's still increasing but at a decreasing rate (*tracing successive equal arc lengths, writing i.d.*)…Uh, then when she hits the top, at seventy-two, it's decreasing at an increasing rate (*tracing successive equal arc lengths, writing d.i.*)…And then when she hits thirty-six feet again it's still decreasing (*making one long trace along the arc length*), but at a decreasing rate (*tracing successive equal arc lengths*, *writing d.d.*).

KM:Ok, so in terms of this one, this quadrant (*pointing to the bottom right quadrant*), could you show me on there how you know it's increasing at an increasing rate? Just show using the diagram...

Zac:So like, a, she moves that much there (*tracing an arc length beginning at April's starting position*), that much here (*tracing an arc of equal length over the last portion of April's path in that quadrant*), uh, the vertical distance there changes by that much (*tracing vertical segment on the vertical diameter*), which is really hard to see with this fat marker. And then, uh, the vertical distance here changes by that much (*tracing vertical segment from the starting position of the second arc length*), which is a much bigger change.

Consider a Ferris wheel with a radius of 36 feet that takes 1.2 minutes to complete a full rotation. April boards the Ferris wheel at the bottom and begins a continuous ride on the Ferris wheel. Sketch a graph that relates the total distance traveled by April and her vertical distance from the ground.

Figure 7. Ferris Wheel Problem

Zac's description suggests reasoning about how the vertical distance (from the ground) and a traversed arc length vary in tandem. In addition to describing both the directional behavior of the vertical distance and the rate of change of the vertical distance with respect to a varying distance traveled (lines 1-12), Zac compares *changes* of vertical distance corresponding to equal changes of distance traveled. Zac followed this interaction by creating a correct graph and describing the graph in compatible ways with Excerpt 4, suggesting that he conceived of the graph as reflective of his constructed covariational relationship. Also important to note is that Zac imagined traveling along a measurable arc length when describing how the quantities varied in tandem, an action that might have been supported by angle measure understandings rooted in reasoning about arcs.

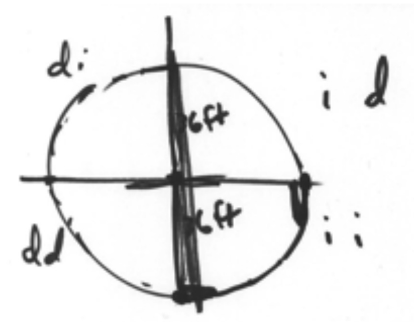

Figure 8. Zac's diagram

Zac's actions also suggest that he imagined the covariational relationship between the relevant quantities as having certain properties over entire arc length intervals. Castillo-Garsow (2010) revealed that students' attentiveness to how quantities covary over a continuum of values is a key aspect of covariational reasoning. In Zac's case, he did focus on comparing completed intervals of variation, but he also showed signs (e.g., tracing continuously along arc lengths while giving his description) of remaining aware that the variation that occurred within each of these intervals held the same properties as the intervals he was comparing.

As an example of a student not explicitly considering variation within intervals, Bob graphed the relation y = sin(3*x*) by first determining *y* values that corresponded to *x* values of 0, $\frac{\pi}{2}$, and π. He plotted these points, claimed, "I guess it just reflects [sin(*x*)]," and then connected the points (Figure 9). Bob did not consider how varies over the interval of $0 < x < \frac{\pi}{2}$ until *after* he plotted the points. He instead used the plotted points to conclude how varies over this interval, as opposed to imaging how sin(3*x*) varies as *x* continuously increases from 0 (sin(3*x*) first increases). Bob's graph stemmed from a pointwise focus and then *filling in* variation between these points, whereas Zac's graph emerged from imagining variation over a (for the most part) continuum of values. Even after being asked for the *y* value that corresponds to *x* = 1 and his stating that the *y* value is sin(3), Bob denoted this on his produced graph as opposed to identifying that sin(3) was a positive value.

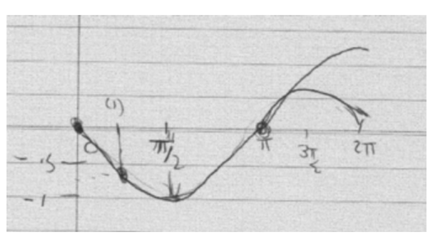

Figure 9. Bob's Graph

Concluding Remarks

The multiple contexts of trigonometry can be thought of as making trigonometry a complex topic, but this complexity also creates an opportunity to support meanings grounded in quantitative reasoning. For instance, the unit circle offers a setting to develop the sine and cosine functions as representative of covariational relationships (Except 4). A circle context also enables using measurement ideas to connect angle measure, the unit circle, trigonometric functions, and the commonly presented trigonometric ratios.

Many of the ideas explored in this chapter aren't specific to trigonometry and involve topics introduced many years previous to students' first experience with trigonometric functions. The above examples include measurement, angle measure, function, graphing, and giving meaning to calculations. These ideas are relevant to an abundance of topics in mathematics education at the K-12 level, which emphasizes that coherence is best thought of in terms of a system of meanings and how these meanings are (or are not) supportive of each other across the mathematics landscape. If we, as educators, expect students to generate coherent mathematical understandings, then it is necessary that we articulate meanings and ideas that are developmentally coherent. We must also gain insights into how such meanings and ideas can be fostered when working with students, which often requires modifying our notions of coherence as we work with students. For instance, it was not until working with a group of pre-service teachers that I became aware of the importance of magnitude-measure reasoning in the teaching and learning of trigonometry. Quantitative reasoning offers a lens for articulating mathematical meanings and systems of ideas. Additionally, quantitative reasoning provides a lens to characterize student thinking and make instructional decisions based on these characterizations. As each student's mathematics is peculiar to that student, it is necessary that all pursuits to create a coherent mathematical experience for students remain attentive to those who are truly doing the creating – the students.

Acknowledgements. I thank David Bressoud and Pat Thompson for laying much of the groundwork in the area of trigonometry. Many of the ideas presented in this chapter stem from their extensive discussions of the teaching and learning of trigonometry, and I am indebted to them for such contributions.

References

Akkoc, H. (2008). Pre-service mathematics teachers' concept image of radian. *International Journal of Mathematical Education in Science and Technology, 39*(7), 857-878.

Bressoud, D. M. (2010). Historical reflections on teaching trigonometry. *Mathematics Teacher, 104*(2), 106-112.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Castillo-Garsow, C. C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth*. Ph.D. Dissertation. Arizona State University: USA.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(66-86).

Moore, K. C. (2009). *An investigation into precalculus students' conceptions of angle measure.* Paper presented at the Twelfth Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference, Raleigh, NC: North Carolina State University.

Moore, K. C. (2010). *The role of quantitative and covariational reasoning in developing precalculus students' images of central concepts of trigonometry*. Ph.D. Dissertation. Arizona State University: USA.

Moore, K. C. (submitted). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*.

Moore, K. C., LaForest, K. R., & Kim, H. J. (in press). *The unit circle and unit conversions.* Paper presented at the Fifteenth Annual Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) Conference, Portland, OR: Portland State University.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Washington, D.C.: Mathematical Association of America.

Reed, S. K. (2006). Does unit analysis help students construct equations? *Cognition and Instruction, 24*(3), 341-366.

Smith III, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.

Steffe, L. P. (1991a). *Composite units and their schemes*. Paper presented at the Annual Meeting of the American Educational Research Association, Chicago, IL.

Steffe, L. P. (1991b). Operations that generate quantity. *Journal of Learning and Individual Differences, 3*(1), 61-82.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge.* New York: Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillside, NJ: Erlbaum.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Thompson, P. W. (1989). *A cognitive model of quantity-based algebraic reasoning*. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: SUNY Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 45-64). Morélia, Mexico: PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^* (pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415-432.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Topçu, T., Kertil, M., Akkoç, H., Yilmaz, K., & Önder, O. (2006). Pre-service and in-service mathematics teachers' concept images of radian. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 281-288). Prague: PME.

Wildi, T. (1991). *Units and conversion charts: A handbook for engineers and scientists*. New York: IEEE Press.