PART I:

INTRODUCTION

QUANTITATIVE REASONING IN CONTEXT

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Abstract

Quantitative reasoning is a complex concept with many definitions and a diverse account in the literature. The purpose of this article is to establish a working definition of quantitative reasoning within context and summarize research on its key processes. The processes of quantitative reasoning include the act of quantification, quantitative literacy, quantitative interpretation of a model, and quantitative modeling. Context underlies all quantitative reasoning, for this review mathematics and science will serve as the context. Finally, the review will explore learning progressions as a framework for establishing learning trajectories for quantitative reasoning.

Quantitative Reasoning in Context
What is Quantitative Reasoning in Context?

The ability to think quantitatively is essential for a citizen of a democracy, allowing them to make informed decisions at home, in the workplace, and on complicated national and international issues that impact their local community. Yet it is hard to define exactly what quantitative reasoning (QR) is or even determine what it should be called. The capacity to reason quantitatively within real-world contexts which impact our lives has many names, including: numeracy, number sense, deductive reasoning, mathematical literacy, quantitative literacy, problem solving, contextualized mathematics, mathematical modeling, and quantitative reasoning. Below are several definitions of quantitative reasoning and quantitative literacy:

- Quantitative literacy involves sophisticated reasoning with elementary mathematics more than elementary reasoning with sophisticated mathematics (Steen, 2004).
- Environmental problems can be better understood using number sense, basic algebra, simple models, and introductory statistics. Quantitative reasoning requires elementary mathematical concepts and techniques used in sophisticated ways (Langkamp & Hull, 2007).
- Quantitative literacy is an aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem-solving skills that people need in order to engage effectively in quantitative situations arising in life and work (International Life Skills Survey, 2000).
- Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgments and to engage in mathematics in ways that meet the needs of that individual's current and future life as a constructive, concerned, and reflective citizen (Programme for International Student Assessment, PISA, 2000).
- Quantitative reasoning is the application of mathematical concepts and skills to solve real-world problems. In order to perform effectively as professionals and citizens,

students must become competent in reading and using quantitative data, in understanding quantitative evidence and in applying basic quantitative skills to the solution of real-life problems (Hollins University, 2011).

- Quantitative reasoning is the power and habit of mind to search out quantitative information, critique it, reflect upon it, and apply it in public, personal and professional life (National Numeracy Network, 2011).
- Numerical abilities that equip students with the capacity to understand and explain the world in quantitative terms; to interpret numerical data; and to evaluate arguments that rely on quantitative information and approaches (BYU Quantitative Reasoning Foundation Document, 2003).
- Beyond arithmetic and geometry, quantitative literacy also requires logic, data analysis, and probability. It enables individuals to analyze evidence, to read graphs, to understand logical arguments, to detect logical fallacies, to understand evidence, and to evaluate risks. Quantitative literacy means knowing how to reason and how to think (Kolata, 1997).
- Quantitative literacy is the ability to interpret and reason with quantitative information, information that involves mathematical ideas or numbers. Quantitative reasoning is the process of interpreting and reasoning with quantitative information (Bennett & Briggs, 2008).

These definitions have some common threads, such as the use of mathematics and statistics within a context and sophisticated reasoning with elementary mathematics, but they also have significant differences. The first two emphasize basic mathematics being used in sophisticated ways, while others include more advanced mathematics and critical thinking. For some the focus is on use in making personal decisions, while others broaden this to making citizenship decisions about global issues.

In addition there is consideration of the act of quantification itself, the mathematical process by which one moves from context to quantity and back to context. Thompson (2011) defines quantification as the process of conceptualizing an object and an attribute of it so that the attribute has a unit measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit. Quantification requires conceptualization and reconceptualization in relation to each other of the object being quantified, the attributes of that object, and the measure of the attribute. The process of modeling and interpreting models is also gaining prominence and should be given consideration as an important component of quantitative reasoning. *Taking Science to School* (Duschl, Schweingruber, & Shouse, 2007) makes the call to move learning towards literacy and modeling practices in the sciences. This National Research Council report identifies four proficiencies in science that all students should attain:

Strand 1: Know, use, and interpret scientific explanations of the natural world
Strand 2: Generate and evaluate scientific evidence and explanation
Strand 3: Understand the nature and development of scientific knowledge
Strand 4: Participate productively in scientific practices and discourse

The four strands emphasize a move from science as inquiry to science practices rooted in model-building and model-refining; moving science out of its current silos of biology, chemistry, earth systems, and physics into a more integrated STEM (Science, Technology, Engineering, Mathematics) approach focused on the application of science in real-world contexts.

Despite the focus of quantitative reasoning on applying mathematical skills and analysis of data through statistical processes, it is not the same as mathematics or statistics. In the seminal work *Mathematics and Democracy: The case for quantitative literacy,* Steen (2001) states:

> Quantitative literacy is more a habit of mind, an approach to problems that employs and enhances both statistics and mathematics. Unlike statistics, which is primarily about uncertainty, numeracy is often about the logic of certainty. Unlike mathematics, which is primarily about a Platonic realm of abstract structures, numeracy is often anchored in data derived from and attached to the empirical world. (p. 5)

The need for quantitatively literate citizens arose in the late twentieth century as numbers became the dominant form of acceptable evidence in socio-political arenas, exposing a public which lacks the appropriate QR skills (Steen, 2001). Steen identifies components of quantitative literacy which citizens should acquire, including confidence with mathematics (numeracy, estimation), cultural appreciation of mathematics (nature and history), interpreting data, logical thinking, making decisions, using mathematics in context, number sense, practical computation skills, prerequisite knowledge of algebra, geometry, statistics, and symbol sense.

Shavelson (2008) in *Reflections on Quantitative Reasoning: An assessment perspective* seeks a definition of quantitative reasoning by exploring three approaches to the topic: psychometric (behavioral roots), cognitive (mental process roots), and situative (social-contextual roots). The psychometric tradition has reached a consensus that there is a QR factor, that is, performance on QR tests are distinguishable from performance on other mathematics tests. QR requires reasoning based on mathematical properties and relations, with a low demand on computation and high demand on reasoning with numbers, operations, and patterns. Shavelson found cognitive research employing interviews and the think-aloud technique which focused on what kinds of reasoning processes are brought to bear in responding to QR type tasks to be lacking. Situative researchers view QR within a community of practice:

> …those individuals engaged in culturally relevant activities in which reasoning quantitatively is demanded and the various resources of the community would be brought to bear on those activities. They would view a person accomplished in QR as having the capacity to engage others in working together to think critically, reason analytically and to solve a problem, for example. Cognitive abilities, from this perspective, reside in a community of practice. (Shavelson, 2008, p. 8)

Madison (2006) provides a situative definition of QR as carried out in real-life, authentic situations; its application is in the particular situation, one dependent upon context including socio-politics. QR problems are deeply contextualized, ill defined, open-ended, real-world tasks that require analysis, critical thinking, estimation, interdisciplinary approaches, and the capacity to communicate a solution, decision, or course of action clearly in writing. Table 1 from Shavelson (2007) contrasts mathematics and QR. The Mathematical Association of America's notion of QR (MAA, 1998) includes interpreting models, using multiple representations (symbolic, visual, numeric, graphic), applying arithmetical, algebraic, geometric, and statistical methods, estimating to determine reasonableness, and recognizing limits of algorithmic methods.

To this point I, the first author, have focused on quantitative reasoning within a context other than mathematics, so some clarification may be needed. In fact Table 1 may be interpreted as falsely suggesting QR is something separate from mathematics or that it is not a vital part of mathematical thinking and reasoning for mathematicians. I interpret the table as laying out the difference between mathematics done abstractly without a real-world context versus math-

Table 1: Contrast of Mathematics with QR

Mathematics	**Quantitative Reasoning**
Power in abstraction	Real, authentic contexts
Power in generality	Specific, particular applications
Some context dependency	Heavy context dependency
Society independent	Society dependent
Apolitical	Political
Methods and algorithms	Ad hoc methods
Well-defined problems	Ill-defined problems
Approximation	Estimation is critical
Heavily disciplinary	Interdisciplinary
Problem solutions	Problem descriptions
Few opportunities to practice outside the classroom	Many practice opportunities outside the classroom
Predictable	Unpredictable

ematics driven by a context of importance to an individual. This does not predicate that QR is outside the domain of mathematics or that all mathematics is well-defined or predictable. Surely the act of quantification we explored above is essentially mathematical in nature. Without this mathematical act quantitative reasoning does not occur. So I do not argue that QR cannot occur without a real-world context, in fact several of the articles in this monograph on the act of quantification will do precisely that. But I want to focus intently on quantitative reasoning within context, and in this article within the context of science. I want to explore the intensively interdisciplinary STEM nature of QR as an integrating factor when used within context. I believe that it is within real-world context that QR transcends both mathematics and the context.

So we now provide our definition of quantitative reasoning within context, (QRC if you will) derived from the literature above as well as our own work focused on QR in STEM and discuss our theoretical framework for QRC.

> **Quantitative reasoning in context** (QRC) is mathematics and statistics applied in real-life, authentic situations that impact an individual's life as a constructive, concerned, and reflective citizen. QRC problems are context dependent, interdisciplinary, open-ended tasks that require critical thinking and the capacity to communicate a course of action.

For the remainder of this paper we will refer to quantitative reasoning in context (QRC) simply as quantitative reasoning or QR. While acknowledging that QR does occur in abstract mathematical situations without real world context, the focus of this paper is on QR in context. We choose as our context for this discussion the area of science, since this is the area in which our work is being conducted and it allows us to focus our discussion of the broad topic of QR.

We propose that quantitative reasoning has four key processes:

- Act of Quantification: mathematical process of conceptualizing an object and an attribute of it so that the attribute has a unit measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit

- Quantitative Literacy: use of fundamental mathematical concepts in sophisticated ways for the purpose of describing, comparing, manipulating, and drawing conclusions from variables developed in the act of quantification
- Quantitative Interpretation: ability to use models to discover trends and make predictions, which is central to a person being a citizen scientist who can make informed decisions about issues impacting their communities
- Quantitative Modeling: ability to create representations to explain a phenomena and revise them based on fit to reality

These processes interact within a quantitative reasoning cycle (Figure 1). When an individual reflects upon a real-life authentic situation that impacts their community or their personal life, they will likely begin reasoning about the situation using a qualitative science account of the phenomena. This qualitative account may be based only on a personal discourse (personal experiential theory of the world), rise to the level of including a school science discourse (acquired knowledge of science often without deep understanding), and perhaps progress to a full scientific discourse (science principles explain the phenomena). Even at the full scientific discourse level the individual may not have engaged in quantitative reasoning; in fact they may have actively avoided using QR. A quantitative account should be sought to provide data-driven support for the qualitative account and to provide evidence supporting conclusions.

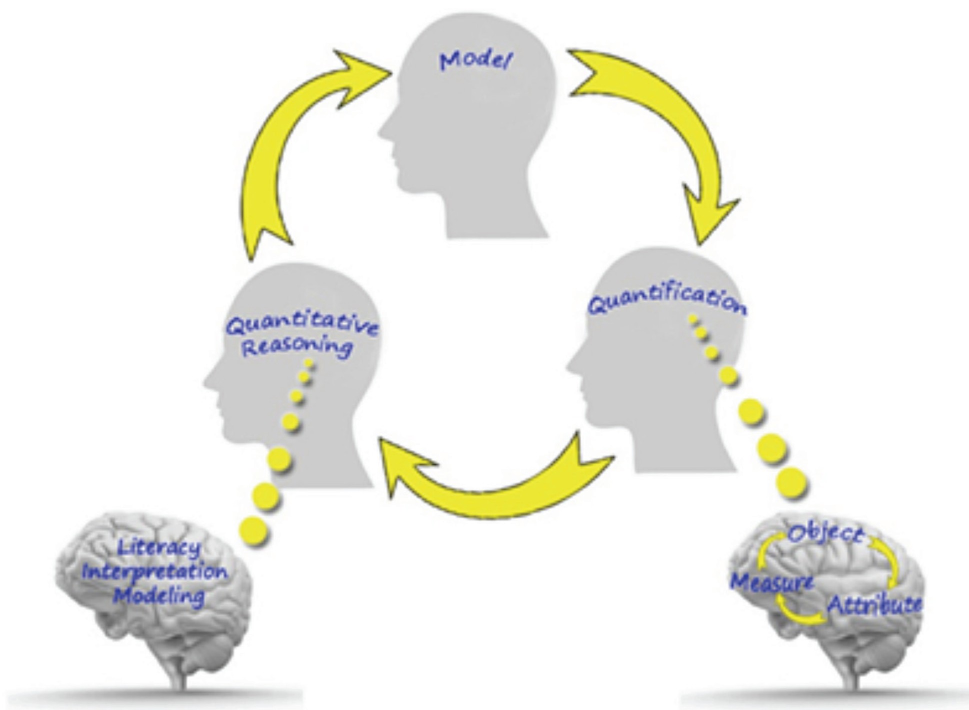

Figure 1: Quantitative Reasoning Cycle

The quantitative reasoning cycle begins with the individual engaging in the act of quantification by identifying objects, their attributes, and assigning measures. Quantification provides variables that can be operated on mathematically or statistically. Second, depending on both the query of interest to the individual and the data they access, they engage in quantitative reasoning through one of the three processes of quantitative literacy, quantitative interpretation, or quantitative modeling. These three processes are interconnected and typically engaging in one requires elements of another. Table 2 explicates the three processes of quantitative literacy, quantitative interpretation, and quantitative modeling, by providing a curricular analysis of QR based on our work in the area of environmental science. The table is not exhaustive, for example it does not include geometric analysis, but lists what we have seen arise as mathematical and statistical tools of import in the context of environmental science. The output of QR is often

***Table 2*:** QR Processes

Processes	**Quantitative Literacy**	**Quantitative Interpretation**	**Quantitative Modeling**
Components	Numeracy • Number Sense • Small/large Numbers • Scientific Notation • Logic Measurement • Accuracy • Precision • Estimation • Units Proportional Reasoning • Fraction • Ratio • Percents • Rates/Change • Proportions • Dimensional Analysis Basic Prob/Stats • Empirical Prob. • Counting • Central Tendency • Variation	Representations • Tables • Graphs/diagrams • Equations •Linear •Quadratic •Power •Exponential • Statistical displays • Translation Science diagrams • Complex systems Statistics & Probability • Randomness • Evaluating Risks • Normal Distribution • Statistical Plots • Correlation • Causality • Z-scores • Confidence Intervals Logarithmic Scales	Logic • Problem Solving • Problem Formulation Modeling • Normal Distribution Model • Regression Model •linear •polynomial •power •exponential •logarithmic • Logistic Growth Model • Multivariate Model • Simulation Model • Scientific Diagram • Table & Graph Models Inference • Inference • Hypothesis Testing • Practical Significance

a model that is then applied within the context of the situation to answer a question. The model may be a loosely connected set of relationships, a table, a graph, a science systems model, or even an analytic mathematical function. This model will need to be tested against the real-world situation and probably modified, leading back to the need to further quantify, and so the cycle repeats.

Quantitative Literacy (QL) underlies both the interpretation and building of models. The variables resulting from quantification are operated on through QL, including such basics as performing computations, comparing, and estimating. QL is mostly arithmetic in nature, epitomizing the sophisticated use of basic mathematics. Within the quantitative literacy process we have identified four major components (Table 2) that underpin the sciences: numeracy, measurement, proportional reasoning, and descriptive statistics and basic probability. There is a great deal of variation in definitions of numeracy, from a mastery of arithmetic symbols and processes to being equivalent with quantitative literacy. We define numeracy as the ability to reason with numbers. Numeracy then is the logic and problem-solving aspect of QR on the arithmetic level. Numeracy includes having number sense, mastery of arithmetic processes (addition, subtraction, multiplication, and division), logic and reasoning with numbers, orders of magnitude, weights and measures. Number sense is defined as awareness and understanding about what numbers are, their relationships, their magnitude, the relative effect of operating on numbers, including the use of mental mathematics and estimation (Fennel & Landis, 1994). Number sense includes the concepts of magnitude, ranking, comparison, measurement, rounding, degree of accuracy, and estimation. Measurement is of central importance to science, so it is separated out from numeracy in the framework as a second component of quantitative literacy which includes accuracy, precision, estimation, and measurement units. Proportional reasoning is often a major conceptual barrier to students, inhibiting their ability to reason quantitatively in science. Here we include pre-proportional reasoning skills such as an understanding of fraction, ratio, percent, and rates, all leading up to proportions. Basic probability and descriptive statistics are essential in data analysis in the sciences and require mostly arithmetic processes, so they are included as a fourth quantitative literacy component.

Quantitative Interpretation (QI) expands upon the use of arithmetic processes in QL to include more mathematically sophisticated algebraic, geometric, statistical, and calculus processes. The QL focus on discovering relationships between variables is supplanted in QI by interpreting models to explore trends and make predictions, a skill that is essential for the citizen scientist. We will use the term model in its broadest sense, certainly as much more than the typical mathematical interpretation of model as a function representing a situation. Models for us include any representation of data and data relationships which allows for interpreting a distinct case within context, exploring trends, drawing inferences, and making predictions. QI entails interpreting models represented as tables, graphs, statistical graphical displays, equations, and complex scientific diagrams, as well as being able to translate between models. It includes the ability to interpret data using probability (randomness, evaluating risks) and statistical analysis (normal distributions, correlation, causality, z-scores, confidence intervals). Algebraic techniques such as logarithms are included under QI since they are common in science and provide a means of interpreting very small or very large scales. QI at the most basic level is the ability to interpret the relationship between two variables; to interpret a model at a given instance or point. At the intermediate level it entails being able to identify trends, to interpret change. At the upper level it is the ability to make predictions through interpolation and extrapolation, to see correlations between data sets, to explain covariation between two variables, and to determine not only the direction but strength of association.

Quantitative Modeling (QM) is inexorably interconnected with quantitative interpretation, for surely when we create models we do so to interpret them. But QM extends QI by requiring the individual to create the model rather than interpret one that is given. We are defining QM as the act of model creation or model generation, while acknowledging that constructed thinking about or with a model is an essential process in model building. We assign this act of constructed thinking about or with models to QI. It could be argued that a citizen will not often create their own models, but extending existing models to answer new questions and understanding how models are created are essential to quantitative reasoning. QM requires a high level of reasoning, including logical thinking, problem solving, hypothesis testing, and caution in making over generalizations. QM engages individuals in formulating problems, developing linear, power, exponential, multivariate, and simulation models, and creating table, graph and even scientific diagram models. QM includes statistical hypothesis testing and understanding practical significance. Duschl's view of science as model building and model testing is underpinned by the ability to quantitatively model a phenomenon.

Above we have provided an overview of what quantitative reasoning in context (QRC) is and established a framework for it. Let us now ground the framework in the literature. The following four sections will expound upon the four key processes of the quantitative reasoning framework: the Act of Quantification, Quantitative Literacy, Quantitative Interpretation, and Quantitative Modeling.

Act of Quantification

An underlying cognitive attribute of QR is the process of quantification, which serves as a process underpinning QL, QI, and QM. Recall that Thompson (2010) defines quantification as the process of conceptualizing an object and an attribute of it so that the attribute has a unit measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit. But what does quantification look like?

Thompson and Saldanha (2003) provided an example of quantification as a root of quantitative reasoning by considering the question: What is torque and how might one quantify it? The object one conceives of as torque is a system involving turning something around a fixed point that behaves differently the farther from the fixed point you are. The attribute of that object is the "amount of twist", as in the case of recognizing it takes more strength to hold a pail of water farther from our body then closer. The measure of torque associated with this attribute is more complicated, since the measure must take into account simultaneously the distance that a force is applied from the fixed end, the amount of force being applied, and that amount of twistiness is proportional to each of these components. Quantification is known to be a significant component in modeling and has been found to be difficult for students (Thompson, 2011). Thompson just completed a National Science Foundation funded research project, Project Pathways: Opening Routes to Math and Science Success for all Students, examining the professional development of secondary mathematics teachers. The project clearly showed the importance of teachers having a productive image of good student reasoning, including an image of how their actions can influence student thinking.

The act of quantification is the conceptualization process in which quantities are assigned to attributes, with properties and relationships formed among them. This process does not focus on a numerical solution, rather on the conceptual process of solving a given problem. Thompson (2011) provided evidence of the act of quantification within a science context with an 8th grade class of students by addressing how they might measure the explosiveness of a grain silo. Collectively the students were quantifying the problem by first thinking that an explosion occurs when flames burn fast and other knowledge they recalled from science regarding oxidation. The students then started to discuss volume of grain dust in relation to surface area exposed, eventually concluding that they need a unit measure of dust surface area per dust volume per silo volume as a way to measure explosiveness. This process required students to quantify the problem by conceptualizing attributes and how they would measure it (Thompson, 2011).

Part of the conceptualization process of the act of quantification is the ability to conceive of the problem mentally through an image. A study conducted by Moore, Carlson and Oehrtman (2009) shows the necessity of a correct mental image in order to quantify the problem and create relationships between the attributes. The study conducted with pre-calculus students showed that when students undergo the process of quantification, they create mental images using drawings or physical objects to represent a given problem. Once students were able to create the correct image, they were able to create correct formulas for solving a given problem. On the other hand, without this mental model, the students found no meaning in the formulas (Moore et al., 2009).

The ability to quantify the problem is necessary if students are to find meaning in numerical computations rather than memorizing formulas which they never derived. Smith III and Thompson (2007) give examples of this process by comparing a numerical solution to a given problem with a quantitative or conceptual solution. Consider the following problem: A father will be 38 years old at some point when he will also be 3 times as old as his daughter, who is currently 7 years old. How old he is right now? The act of quantification takes problems such as this and emphasizes mathematical reasoning when solving problems by focusing on quantities and relationships among them (Smith & Thompson, 2007). In this case, the difference between the two people's ages and the ratio in how many times older the father is than the daughter would be the relationships of focus. Having created these quantities and their relationships, students are able to develop a conceptual understanding of the given problem, and in turn use mathematical concepts to solve the problem. These conceptual ideas create a support for using algebra as a tool for problem solving.

An important aspect of quantification is covariational reasoning, which is defined as "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p.354). Lobato and Siebert (2002) conducted a teaching experiment in a course on linear functions and slopes in which they closely studied the progress of one particular student. The focus was on a wheelchair ramp problem where the students are questioned regarding the steepness of the ramp. In particular, Lobato and Siebert believed that focusing on slope as a calculation to determine steepness would cause some loss in potential transfer of knowledge, rather the focus should be on slope as a ratio of two varying quantities measuring some attribute, that is on covariation. The student first believed that height was the only determining factor of steepness. After some probing from the interviewer, the student was able to accept that length also influenced the steepness of the ramp, but the student could not vary the height and length of the ramp independently. However, by the end of the teaching experiment, the student was able to construct a ratio between the height and length. This is an example of the development of covarying quantities. It is important to take note that when the student was asked if he had learned about slope, he responded that it is rise over run, however he did not interpret slope to be a measure of steepness of the ramp. This indicates that the attributes of the object must first be determined and a relationship formed between the attributes, before measurement is performed.

An earlier case study by Thompson (1994) also showed the obstacle students have in understanding proportional reasoning or ratio due to lacking an understanding of covariational reasoning. The student in Thompson's study thought of time as an implicit quantity with respect to distance, in a given speed problem. Just as in the Lobato and Seibert study, the student was not able to see two quantities of equal stature in the beginning of the teaching experiment. However, in both cases, the students were able to create the ratio by the end of the experiment by first acknowledging two explicit quantities. In this case the student acknowledged time as an explicit quantity and in turn created the ratio. The traditional teaching of speed as distance divided by time does not allow for the development of the concept of speed as a ratio and has little relevance to the understanding of speed (Thompson, 1994).

Quantitative Literacy

We propose four components of QL related to science: numeracy, measurement, proportional reasoning, and descriptive statistics/basic probability. Science requires students to work with very large and very small numbers, a numeracy skill. Putting these numbers in perspective is a challenge for students. For example the diameter of a hydrogen nucleus is approximately 0.000000000000001 meter while the total energy consumption in the United States is 100,000,000,000,000,000,000 joules. Scientific notation is used to represent small and large numbers in a format that is easier to comprehend and easier to perform operations upon: hydrogen nucleus is 1×10^{-15} meters and U.S. energy consumption is 1×10^{20} joules. This facilitates issues of order of magnitude or powers of 10 in science, such as U.S. energy consumption being 35 orders of magnitude of a hydrogen nucleus. It also leads to the need to do arithmetic operations with powers of 10, such as multiplying powers of 10 by adding the powers. Science is replete with extremely small and large numbers which are often incomprehensible to students. Three techniques for bringing numbers into perspective are estimation, comparisons, and scaling (Bennett & Briggs, 2008).

The sciences require careful comprehensive measurement of quantities such as distance, area, volume, discharge (1 acre-foot per day), mass, density, force, pressure, work, moment, energy, power, and heat. Some quantities are measured directly (i.e. length in meters or feet, weight in Newtons or pounds, and temperature in Celsius, Fahrenheit, or Kelvin) using a variety of tools such as rulers, scales, inclinometers, spectrometers, and fluorometers, while others are measured indirectly or a calculated from other measures (i.e. area, volume, stream discharge (volume/time), density (mass/volume), force (mass x acceleration), pressure (force/area), work (force x distance), and power (work/time)). Fundamental characteristics of measure are accuracy (how close the measurement is to the actual value), precision (how refined the measure is), and error (Langkamp & Hull, 2007). Error in a measure can be calculated as an absolute error or absolute change (measured value – true value) or relative error or relative change (ratio of absolute error/true value). Estimation of measures, especially those that are calculated, provide an independent check on the calculations. Wiser and Smith (2009) conjecture a learning progression in which students move from a conception of felt weight to quantifying weight as measured by a scale. This progression requires understanding the quantitative nature of measure. They propose that students move from measuring weight and volume to an understanding of density by graphing weight as a function of volume.

Proportional reasoning encompasses complex cognitive abilities which include both mathematical and psychological dimensions. It requires a significant conceptual shift from concrete operational to formal operational levels of thought (Piaget & Beth, 1966). It has been proposed as a major barrier to students' development of mathematical understanding, as well as negatively impacting the development of scientific understanding. Its pivotal position in science is as the most common form of structural similarity, a critical aspect of recognizing similar patterns in two different contexts. In addition, proportional reasoning underpins many of the

QL components, including measurement, numeracy, and dimensional analysis. Proportional Reasoning is a "form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (Lesh, Post, & Behr, 1988). Lesh, Post, and Behr consider the essential characteristic of proportional reasoning to involve reasoning about the holistic relationship between two rational expressions (fractions, quotients, rates, and ratios). Proportional reasoning is not the ability to employ the cross multiplication algorithm, in fact rote use of this algorithm often replaces such reasoning. Early phases in proportional reasoning involve additive reasoning (A – B = C – D) and multiplicative reasoning (A x B = C X D). Traditional proportional reasoning involves relationships of the type A/B = C/D, where one of the values is unknown. Karplus et al. (1983) views proportional reasoning as a linear relationship between variables such as y = mx, where the y-intercept is 0. Proportional reasoning requires students to first understand fraction a/b, which at the most basic level is interpreted by students as comparing the part (numerator a) to the whole (denominator b) for like quantities. This basic concept of fraction underpins one notion of percentage as comparing part-to-whole. This is also an example of using a percentage to describe change.

Ratio represents a relationship between two different quantities, focusing on part-to-part comparisons. Students must attain the conception of ratio before being able to set up equivalent ratios, one of the fundamental conceptions of proportional reasoning. The student can use the concept of ratio and multiplication to perform dimensional analysis in science. Dimensional analysis is the tracking of units when performing calculations or performing unit conversion. Performing dimensional analysis requires the student to understand ratios. Literally a student can track the calculations they need to perform by tracking the units. Other examples of ratios in science include parts per thousand or million, conversion factors such as 1000 cm^3/1 liter, and normalizing data. Percentage is a fraction or ratio represented as a part of a 100. Percentages can compare part-to-whole which is a representation of the fraction conception. Percentages can also compare part-to-part for like quantities, which is a representation of the ratio conception. Another use of percentages in science is making comparisons between a comparison value and a reference value using the percentage difference formula.

Students develop conceptions of rate of change separate from ratio, with the most basic conception of change that of a constant rate found in linear models. This change is represented by the slope of the line. Many students may conceive of change as being univariate at first, that is they will take the difference of two occurrences of one variable without accounting for the change in a second. Slope is bivariate in nature, it takes into account the change in one variable with respect to another. This is the covariational conception discussed in the act of quantification section above.

A proportion is an equivalence between two ratios: a/b = c/d. Many students can manipulate proportions to find the missing value, as in 2/5 = x/10, however this may indicate only rote use of the cross multiplying algorithm. True proportional reasoning requires a perception of structural similarity; a conception of n times as many. If a student reduces A/B = C/D to P = C/D when solving, then they are not using possible structural relationships but are solving using algebra without regard to structure.

At a higher level of proportional reasoning a student moves from a conception of proportional reasoning as equivalent ratios to a conception of linear direct variation y=kx. Science contexts for proportional reasoning are often in the linear direct variation form. In addition students may extend this to indirect variation y=k/x. This requires an understanding of the underlying algebraic concepts of equivalence, variable, and transformations (structural similarity and invariance).

Probability is the chance of occurrence of an event, with the theoretical probability defined as: P(e)=(number of successful outcomes of event)/(total number of possible outcomes). Earth systems cannot be manipulated like dice to determine a theoretical probability. Often scientists can only estimate the probability through observations of the system. Empirical (experimental) probability is determining a probability based on observations or experiments. A probability is always between 0 and 1, so the probability of an event not occurring is 1 – P(e). For example if the probability of having a flood is 26%, then the probability of not having a major flood in a given year is 1 - 0.26 = 0.74 or 74%. Odds for an event are the ratio between the event occurring and the event not occurring. The odds of having a major flood in a given year are (P(e))/(1-P(e))=0.26/0.74 or the odds are about 1 in 3. There are more complicated probability problems such as conditional probability with dependent or independent events which are of use in science. Computing experimental probabilities requires counting the occurrence of events. Combinatorial techniques allow one to count without listing all the events. Some basic counting principles are the multiplication principle, the pigeonhole principle, permutations, and combinations. These methods are based in arithmetic principals, but quickly lead to the use of exponential and factorial principles. Theoretical probability may not often be used in science where the focus is on analyzing observed data, so no more will be said about these counting principles at this time.

Descriptive statistics allow us to summarize and describe data. The fundamental descriptive statistics are measures of the center of a distribution and measures of the spread in a distribution. Measures of central tendency include the mean, median, and mode. The mean is the sum of all the values divided by the number of values, while the mode is the most frequently occurring value. The mode is not often used in analyzing scientific data. The mean and median values for a data set can differ significantly, so it does make a difference what measure of central tendency is reported.

Variation is a measure of how much the data are spread out. The simplest measure of variation is the range which is the difference between the largest and smallest values in the data set. While easy to calculate the range can be misleading, since one outlier can make it appear the data set is more spread then it is. To avoid this one can use quartiles (values that divide the data set into quarters) and the 5 number summary – lowest value, lower quartile, median, upper quartile, and highest value.

The most commonly used measure of variation is standard deviation, which measures the average distance of all data values from their mean. So we find the deviation which is the distance of a value from the mean, square the deviation so that positive and negative deviations don't cancel out when adding them which could conceal spread, and take the average of all deviations by summing them and dividing by one less than the total number of data values (this is an adjustment for working with a sample rather than a population). The result is called variance. But variance is in squared units and our original data is not squared, so we square root the variance to get standard deviation, which is in the same units as the original data. By Chebyshev's Rule for any set of data at least 75% of the data lie within 2 standard deviations of the mean and at least 89% of the data lie within 3 standard deviations of the mean. Any data value that lays 3 or more standard deviations from the mean is called an outlier and it is common practice to discard them from the data set.

The majority of what we have discussed to this point as elements of QL belong to the realm of number and arithmetic, only variation requires algebraic operations of taking roots or powers. So we see the meaning of Steen's admonition that QL is sophisticated reasoning with elementary mathematics. Other basic statistics that are used in science crossover from arithmetic to algebra such as z-scores (number of standard deviations a data point lay above or below the mean) and confidence intervals. But it is amazing what one can do with arithmetic if they can but reason with it within a context.

So QL is the ability to use fundamental mathematical concepts in context and it plays an important role in today's modern society. The media, workplace, and our everyday lives have been filled with quantitative data. It is imperative that everyone be able to interpret and use the data presented to them to make informed decisions (Madison & Steen, 2003; Steen, 2001, 2004). According to the Quantitative Literacy Design Team, "Most U.S. students leave high school with quantitative skills far below what they need to live well in today's society (Steen, 2001, p.1)." Data from the Third International Mathematics and Science Study (TIMMS) revealed that students in the U.S. performed relatively low compared to other countries in their mathematical skills. However, the same students responded that they enjoy mathematics and are confident in their performance (Wilkins, 2000).

Numeracy, which in our case is the ability to reason with numbers, is the key to understanding data in our current society. Understanding requires more than formulas rather it requires the ability to reason and think quantitatively (Steen, 2001). Wilkins extends the definition of QL to one who is willing to "engage in situations that require a functional level of quantitative reasoning" (Wilkins, 2000, p.408). Unfortunately, many students lack QL skills due to a shallow coverage of these concepts in schools, due to a focus on college prep and a narrow curriculum with the singular goal of calculus as the culminating experience. Many of the fundamental QL skills such as measurement, geometry, data analysis and probability do not get much time in the curriculum (Madison, 2003).

The fundamental mathematical concepts necessary for non-calculus track students is addressed by Briggs (2004). Based on a study he conducted on a group of non-calculus track undergraduate students fulfilling general education requirements, Briggs found that the fundamental mathematical knowledge needed were logical thinking skills, estimation, statistical literacy, and financial mathematical knowledge. Students find it difficult to attain these skills as they often do not see the connection between the mathematical knowledge they acquire and applications to their daily lives. QL must be made compelling to students by showing them examples of how it impacts their lives, which are filled with quantitative information (Briggs, 2004). According to Madison, "Many students do not believe that mathematics has very much to do with their everyday lives" (Madison, 2006, p.2325).

One of the fundamental skills in QL is proportional reasoning. Taylor and Jones (2009) conducted a study with middle school students on their proportional reasoning abilities. The students attended a summer camp in which they participated in activities on surface area to volume applications. As a result, Taylor and Jones found that there is a significant correlation between proportional reasoning abilities and surface area to volume relationships. The study shows that there may also be a relationship between proportional reasoning and the scaling concept in science.

A study by Jones, Taylor, and Broadwell (2009) was done on the sense of scale and estimation using the body as a measurement tool. Using the body as a ruler allowed students to better visualize a measure and become more accurate in estimation. Consequently, students who had a better understanding of proportional reasoning also performed better by giving more accurate values of estimation. Proportional reasoning is positively correlated with applying scale and estimation in real world problems. So proportional reasoning can be an obstacle for students in their understanding of concepts in science. Students' sense of scale was also studied by Delgado, Stevens, Shni, Yungker, and Krajcik (2007), who examined students' understanding of how big one object is as compared to another, and the objects absolute size.

Madison (2006) presents a pedagogical challenge related to QL: students have been taught for many years in traditional mathematics classrooms so their habits and attitudes can become obstacles. Thompson (1994) and Lobato and Seibert (2002) showed that teaching students formulas, such as slope is rise over run or speed is distance divided by time, does not provide students with the understanding to apply these concepts. Another difficulty is that in order for students to accept that they should understand a concept they must be engaged. In this case they should be engaged in the material to a greater extent than in a traditional mathematics classroom (Dingman & Madison, 2010; Madison, 2006). Some positive outcomes have arisen in the studies done by Dingman and Madison. They concluded, "One of the positive changes we have seen is the modest shift in the students' views regarding the relevance of the mathematics in their everyday life. By placing the mathematical and statistical topics in real-world contexts, the connections to their life are much more real and apparent than their past experiences in learning mathematics. (Dingman & Madison, 2010, p.6)."

Quantitative Interpretation: A Key Citizenship Aptitude

Quantitative interpretation is the ability to use models to make predictions and determine trends. Due to the fact that a model can take various forms (e.g. tables, graphs, statistical graphical displays, equations, or complex scientific diagrams) issues can arise in two different senses. First, the interpretation of a model can be challenging for students. Second, the translation between models representing the same content can provide a challenge. For example, given a table and a graph of the same data, students can struggle to see the relationship between the two different representations. Understanding the multitude of representations available is important for organizing, synthesizing, explaining, and displaying data, which in turn is essential for being a citizen scientist. Existing research mostly focuses on a specific form of representation and this literature indicates that more research is needed to enhance the understanding of how students' interpret models.

As the American Education Reach Out (AERO) organization clearly states: "Representations are necessary to students' understanding of mathematical concepts and relationships" (2010, p. 11). Early understanding of multiple representations is important for students to progress mathematically (Schwartz & Martin, 2004; Zahner & Corter, 2010). Zahner and Corter (2010) propose in their model that students pass through four stages when problem solving. Stage 2 is mathematical problem representation. According to their model, to reach level 3 and 4, students must pass through this stage first. Therefore, the inability to interpret and represent a problem could be a barrier to student problem solving.

Representations take on numerous forms, from graphs and tables to equations and written text. They also vary in popularity. For example, Lowrie and Diezmann (2009) found maps are a type of representation that has increased in popularity recently. Maps are one representation that requires a certain amount of "decoding", (Logan & Greenlees, 2008; Lowrie & Diezmann, 2009) which can be very challenging for students. They argue that students can encounter difficulties when trying to separate graphical features from other demands posed by the task, such as linguistic knowledge and mathematical knowledge.

Examining multiple representations is important when discussing learning because often graphical representations and text appear side-by-side. Stroud and Schwartz's (2010) base their study of metaphoric graphics in chemistry instruction on the notion of the redundancy effect. This occurs when students become overwhelmed with the amount of information presented as "text-based content" and it interferes with student learning (Stroud & Schwartz, 2010). Thus, knowing how students read, interpret, and process simultaneous representations when learning content is important to consider when developing learning progressions and planning instruction (Clement, Lochhead, & Monk, 1981). This supports our inclusion of the ability to translate between different representations. When more than one representation is presented side-by-side, it is important for the student to make meaning of each representation and draw connections among the different representations.

When asked to classify mathematical problems according to their difficulty many would argue that story problems are among the most difficult for students to solve. According to a study conducted by Koedinger and Nathan (2004) the exact opposite seems to be true. They divide the problem-solving process into two phases, a comprehension and a solution phase (Koedinger & Nathan, 2004). During the comprehension phase the type of problem representation chosen plays a vital role for the students' understanding as well as their problem solving strategies. Koedinger and Nathan (2004) discovered in their study that students showed fewer difficulties concerning story word problems than with symbolic problems, such as equations, when the language and context used are accessible to them.

Friel, Curcio, and Bright (2001) investigated the comprehension of statistical graphs. They identified four categories influencing graph comprehension: purpose for using a graph, characteristics of the tasks, characteristics of the discipline, and the characteristics of graph readers. According to Friel et.al., a vital component of graph comprehension is understanding that there are three areas of graph perception, namely "visual decoding, judgment task, and context" (Friel, Curcio, & Bright, 2001, p.152). All three components need to be adequately addressed to ensure improved graph comprehension. This is also supported by the study conducted by Zahner and Corter (2010). They call for further research to enhance teachers' understanding of how students comprehend graphs, which in turn can result in new instructional strategies.

Thompson has focused his interest multiple times on representations (Oerthman, Carlson, & Thomposon, 2008; Thompson, 1994, 1999, 2002; Thompson & Saldanha, 2000). One aspect Thompson (1994) focused on was the concept of average rate and speed. Students display difficulties when it comes to distinguishing ratio and rate. Thompson (1994) exclaims that "any problem typology suffers this same deficiency, namely that any given situation can be conceived in a multitude of ways" (p.53). During his teaching experiment he discovered that a model can easily be interpreted or viewed in different ways, which emphasizes the subjective character of interpretation. Furthermore, Thompson (1999) discovered that even after spending a significant amount of time in groups working on an interpretation of a specific point on a graph, students' individual answers showed internal incoherence. All groups stated the results of their collaboration on what a point (represented as an ordered pair) on a graph means. Thompson then asked each student to write down an answer individually. These answers differed from their group work; such that more internal incoherence was noted and a large portion of students could not even state an answer. Thompson (1999) concluded that even though the students had reached an agreement on one interpretation within their groups, it did not necessarily mean that each student had reached the same interpretation internally. He states that "[W]hat individuals understand may be expressed as something stable in the way they interact, but the extent to which interaction-as-stable-pattern reflects individuals' understandings may be uncertain at best" (1999, p.6). It indicates the difficulties of assessing students' understanding. It poses problems when interpreting students' answers in term of whether or not the student has reached an understanding of a model or not. Thompson and Saldanha (2000) also studied the role of representations in the field of statistics and probability. They found that student misunderstandings in this area can be introduced as early as elementary school.

Quantitative Modeling

Quantitative modeling is the ability to create representations to explain phenomena and is inseparable from QI. It requires a high amount of logical thinking and reasoning for an individual to produce a model. Just as QI refers to a multitude of representations, QM has many different forms as well (e.g., formulating problems, developing linear, power, exponential, multivariate, and simulation models, and creating table, graph and scientific diagram models).

Why should we care about model building if it is not currently a focus of schools and citizens seldom develop their own models? Taking Science to School (Duschl, Schweingruber, & Shouse, 2007) identifies four proficiencies in science that all students should attain (see introduction for details). The four strands emphasize a move from science as inquiry to science practices rooted in model-building and model-refining; moving science out of its current silos of biology, chemistry, earth systems, and physics into a more integrated STEM approach focused on the application of science in real-world contexts. Among these real-world contexts would be the grand challenges in environment identified by the National Research Council (2001), which offer the opportunity for students to address global issues and their impact on local communities. Science as model-building is defined as learning science as a process of building theories and models using evidence, checking them for internal consistency and coherence, and testing them empirically (Duschl et al., 2007). The NRC also states in *A Framework for K-12 Science Education: Practices, Crosscutting Concepts, and Core Ideas* (NRC, 2011) that STEM disciplines permeate our lives and thus are central to meeting humanity's most pressing challenges. The report proposes three dimensions around which STEM education should be coordinated:

- Crosscutting concepts that unify the study of science through common application across science fields
- Scientific practices
- Core ideas in physical sciences, life sciences, earth and space sciences, technology, and the applications of science

Common to both national reports is a focus on the interdisciplinary study of real-world problems that emphasize key STEM understandings and practices. An important component of this integration is quantitative reasoning, which can serve as an integrating factor of mathematics and statistics into science.

The seminal research done by Schwarz, Reiser, Davis, Kenyon, Acher, Fortus, Shwartz, Hug, and Krajcik (2009) in the Modeling Designs for Learning Science (MoDeLS) project defines scientific modeling as elements of practice including constructing, using, evaluating, and revising scientific models, and the metaknowledge that guides and motivates the practice. Their learning progression for scientific modeling has two dimensions: scientific models as tools for predicting and explaining; models change as understanding improves. This learning progression for science modeling is unique in that the primary focus is a scientific process rather than a scientific concept, which is at the core of most science learning progressions. The progression is focused on upper elementary and middle school grades. Students moved along the learning progression, moving from illustrative to explanatory models, developing increasingly sophisticated views of the explanatory nature of models, and developing more nuanced reasons to revise models. While MoDeLS provides a scientific qualitative account of modeling, we believe it needs to be expanded to include the quantitative account of modeling across grades 6-12.

Lesh (2006), in a paper on students' modeling abilities, calls for a deep understanding of complex systems which are becoming more and more prominent in the 21st century. This refers to nearly any attempt at modeling real world phenomena. Students need to be more familiar on how to focus on the most important processes they want to model, as well as strategies of modeling. Lesh (2006) differentiates between three different kinds of complex systems which can be very challenging for students to understand: real world systems, conceptual systems, and models describing and explaining students modeling abilities (Lesh, 2006). The components of each system are defined and labeled, but they derive their meaning by being part of the system and can therefore not be investigated individually. According to Lesh (2006) mathematics includes the learning of sets of rules to the same degree as it includes the ability to model real world situations (Lesh, 2006). This idea of enhancing the design of tasks that engage students in complex modeling is exemplified in a study by Lesh, Middleton, Caylor, and Gupta (2008). They introduce the idea of model-eliciting activities (MEA) to enhance students' modeling abilities. Their MEA intend to make students' thinking explicit around the use and creation of models.

Thompson's (2011) approach to quantitative modeling focuses on the act of quantification, especially concerning dynamic situations, and quantitative covariation. He defines two aspects that are essential when using mathematics to model dynamic situations. First, students need to understand the quantities themselves and visualize that their images include values that vary. Second, students need to form a representation of the "object made by uniting those quantities in thought and maintaining that unit while also maintaining a dynamic image of the situation in which it is embedded" (Thompson, 2011, p. 27).

QM plays a major role in the sciences (Adúriz-Bravo, 2012; Lehrer, Schauble, Carpenter, & Penner, 2000; Matthews, 2007; Svoboda & Passmore, 2011) and modeling in science needs to be taught in a dynamic manner. Scientists develop, use, and revise their models in a cyclic process. This process should be accessible for students so they understand the dynamic nature of science. Additionally, there is a need for an emphasis on argumentation in science for which mathematical reasoning skills are a prerequisite. Matthews (2007) describes the process as beginning with observations of real objects, which then need to be conveyed linguistically in some way. Within this step is where quantitative reasoning has to take place. This conveyed information, or model, is set within a discourse and can now be scientifically debated. The final step is the revision of the model and then the cycle starts over again. Adúriz-Bravo (2012) describes this process with the words: "inventing, applying, refining, and learning models (Adúriz-Bravo, 2012, p. 16). Another study which supports the importance of modeling for students was conducted by Lehrer and colleagues (2000). They investigated two situations where children learn through design in elementary school. Most important for education by design are the tasks, tools, and representations. The basic idea is to align classroom activities to how scientists in the real world work. It begins with a problem (task) which needs to be specified and should lead to the construction of a working model. This model needs to be tested and tried and if needed re-designed. Finally, students need to elaborate the important ideas behind their model and data. Their two classroom implementations also demonstrated how important it is to connect new material to children's existing knowledge. Children's previous knowledge needs to be assessed by teachers before introducing new tasks.

Doerr and English (2003) investigated how instructional tasks can enhance students' modeling abilities in two middle school classrooms, one in Australia and one in the USA. Their instructional tasks were different from traditional textbook problems and addressed "the creation of ranked quantities, operations and transformations on those ranks, and, finally, the generation of relationships between and among quantities to define descriptive and explanatory relationships" (Doerr & English, 2003, p. 131). They provided instruction which differed from the traditional way of simply guiding students through specific problem solving strategies, enabling them to develop their own ways of approaching, refining, and expanding their thinking about problems. Their results showed that after students were exposed to the novel instruction, they were capable of defining their own quantities (ranks) and operating on those quantities to build their basis for interpreting and revising their models. Additionally, they discovered that since students approached problems with a variety of different models, the communication about and translations between models was facilitated.

Issues with modeling abilities can begin in elementary school as indicated by Verschaffel, De Corte, and Vierstraete (1999). They investigated upper elementary students' modeling ability concerning addition and subtraction problems and discovered many student difficulties. According to them, the majority of students' difficulties are due to the superficial and traditional teaching approaches which focus simply on the process, but do not discuss the appropriateness and relevance of the operation within a context. Students tend to move straight to manipulating with the numbers of a given problem without having checked if their operations make sense contextually. This indicates, as in the study above, that one key to improving students' modeling abilities is through changing instructional strategies.

Lehrer and Schauble (2004) also discovered the need for improving instructional strategies. Their study explored upper elementary students' thinking in the context of natural distribution. Students worked with data to model plant growth. The main emphasis during instruction was placed on investigations carried out by the students. The students developed different models in order to draw inferences about plant growth as well as make predictions for the future. In addition, their understanding of influencing factors, such as light or fertilizer, was enhanced and the effects were incorporated in the models. This study demonstrated how important it is for students to construct their own models of real world phenomena, allowing them to draw inferences and to reach a deeper understanding.

One of the most difficult types of modeling for students is algebraic modeling. In a case study conducted by Izsák (2003) on how students go about modeling a physical device, he discovered that students can develop a set of skills that allows them to construct, evaluate and test their developed equations. This set of skills results from a combination of students' prior common knowledge and carefully guided student-based instruction. The students were able to develop a linear system of equations, evaluate it by plugging in values, and discuss and revise it in collaboration with each other. Hence, algebraic modeling can be mastered by students with the appropriate guidance, but it is not common in current classroom practice.

Thompson (2002) also focused on the role of instructors and how their teaching can be enhanced. He introduced the concept of didactic models, which "is for instructors and instructional designers what they intend students will understand and how that understanding might develop" (Thompson, 2002, p. 212). Thompson's didactic models can be compared to the idea of a learning trajectory. They are intended to provide a path indicating how understanding might develop. The major difference is that Thompson's didactic model shows a "clear separation between descriptions of instructional sequences and descriptions of what students are to understand" (Thompson, 2002, p. 213).

To instantiate science as modeling in the classroom we must move from direct instruction of STEM as a collection of facts to be mastered and from a narrow hypothesis testing view of scientific inquiry, toward curriculum, instruction and assessment models that embrace the four proficiencies strands in *Taking Science to School*. Science teaching would be driven by science as model-building and refinement. This reformulation of curriculum, instruction and assessment proposes a significant change in the current teaching and learning of STEM which is within the purview of the schools. It will require a significant shift in both student and teacher expectations in the classroom. Science from a model-building perspective is best achieved through integrated, interdisciplinary STEM instruction that incorporates place-based (Smith & Sobel, 2010) and problem-based pedagogies (Edelson & Reiser, 2006). Engaging students in real-world problems will require them to bring to bear knowledge and understandings from multiple subject areas, including biology, chemistry, earth systems science, physics, mathematics, and statistics. Building models and testing them will push both the teachers and students capabilities. There is a need to study the potential for students to engage in model-building and testing, establishing trajectories of student development though the creation of learning progressions which can assist teachers in tracking student formative development, and eventually to the construction of professional development programs. The theoretical foundations and pathways so established will guide the creation of developmentally appropriate performance tasks that will provide students with experiences that further their understanding of key concepts across STEM.

QR in Context

A keystone of our definition of quantitative reasoning is that it occurs within a real-world context. The importance of quantitative reasoning in context, especially within the context of science, is supported by research (Clase, Gundlach, & Pelaez, 2010; Eggert & Bögeholz, 2009; Gibson, Callison, & Zillmann, 2011; Hastings et al., 2005; Speth et al., 2010).

A study that links environmental sciences to quantitative reasoning was performed by Eggert and Bögeholz (2009). This quantitative study asked students to make informed decisions in regards to the environment and sustainability by means of quantitative reasoning. A pencil-and-paper test focusing on the students' decision making competence was taken by 370 students. Another 83 students took the test with additional items addressing their verbal skills. The results of their study support the claim that "the first 2 years at secondary school seem to be crucial for fostering decision-making competence with respect to the use of decision-making strategies" (Eggert & Bögeholz, 2009, p. 249).

Speth and colleagues (2010) investigated the importance of quantitative reasoning in a large-enrollment Introductory Biology course at a research university by means of formative and summative assessments. The results of their pre-intervention assessments showed that students were experiencing great difficulties with modeling via graphs and interpreting the information within the context. Their intervention throughout the course led to a significant gain in the students' abilities as well as a better understanding of the science being taught. According to this study quantitative reasoning needs to be incorporated throughout the entire curriculum and quantitative thinking should become "an intrinsic component in the construction of scientific knowledge" (Speth et al., 2010, p. 331).

An article published by Hastings and colleagues (2005) also supports the importance of quantitative reasoning skills in the field of biological sciences. They illustrate five examples of quantitative reasoning in environmental biology, such as evolution of virulence, community ecology of disease, management of renewable resources, large-scale and global ecology, and scaling from individuals to ecosystems (Hastings, Arzberger, Bolker et al., 2005). They define three major themes that are essential for quantitative biology: spatial and temporal variability, statistical integration of theory and data, and the problem of scaling. Stochastics, for example, plays a crucial role for investigating population development. In conclusion, Hastings and colleagues (2005) suggest that quantitative reasoning and modeling support biologists by clarifying environmental challenges and providing a deeper insight. The important role of stochastics in biology is also underlined by two studies investigating the connection between the development of polar bear cubs and the decline of sea ice, as well as the influence of climate change on polar bear populations (Hunter, Caswell, Runge, Regehr, & Amstrup, 2010; Rode, Ammstrup, & Regehr, 2010).

A study conducted by Clase and colleagues (2010) based on the importance of quantitative reasoning in biology calls for a change in teaching biology. In their study Calibrated Peer Review (CPR), a web-based program, was foundational to the teaching method in an undergraduate class at a large land-grant research university. CPR assignments included several quantitative aspects including models and graphs, and students were asked to make causal conclusions and graph relationships. Their results indicate an increase in students' understanding after fulfilling the CPR assignments over one semester (Clase, Gundlach, & Pelaez, 2010).

A study conducted by Gibson et al. (2011) investigates the effect of using statistical representations taken from case studies and news reports on a group of undergraduates with varying quantitative reasoning skills. Students with greater ability in arithmetic skills achieved more accurate results on a QR computer-based test. The test provided students with articles from which they were asked to extract and assess the numerical information in terms of the underlying content, and give estimations of, for example, ratios. They also found that students with lower ability arithmetic skills are "more attentive to, and affectively more engaged by, personalized information provided by detailed case reports" (Gibson, Callison, & Zillmann, 2011, p. 114).

This completes our review of quantitative reasoning in context. We now turn to a discussion of how learning progressions can serve as a mechanism for studying and improving QR in schools.

QR Learning Progressions

The NSF Pathways[1] project is examining how environmental literacy develops as students move through grades 6 to 12. A central component of the project is the development of learning progressions that provide a trajectory through which students pass on their way to becoming environmentally literate citizens. One aspect of this development is the role that quantitative reasoning plays. So what is the interplay between QR and learning progressions?

Taking Science to School (Duschl et al., 2007) recommends that learning and curriculum designs be organized around learning progressions as a means of supporting learners'

development towards attaining the four proficiencies in science. Why learning progressions? The Consortium for Policy Research in Education (CPRE) report *Learning Progressions in Science: An Evidence-based Approach to Reform* (Corcoran, Mosher, & Rogat, 2009) identified learning progressions as a promising model that can advance effective adaptive instruction teaching techniques and thereby change the norms of practice in schools. The CPRE report defines a learning progression as "empirically grounded and testable hypotheses about how students' understanding of, and ability to use, core scientific concepts and explanations and related scientific practices grow and become more sophisticated over time, with appropriate instruction" (p.8).

The hypothesized student pathways to mastery of core concepts are based on research about how students' learning actually progresses. The hypotheses are tested empirically to assess validity of the pathway against actual student progress. The CPRE report infers that "if this work is pursued vigorously and rigorously, the end result should be a solid body of evidence about what most students are capable of achieving in school and about the particular sequence(s) of learning experiences that would lead to proficiency on the part of most students. The NSF Pathways QR Theme team hypothesizes that QR is essential for data-based and modeling approaches to learning the sciences. The team is focused on the theoretical bases that will inform

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the design of learning progressions for quantitative reasoning in science as model-building. The team is developing learning performance assessments that inform the progressions and teachers' adaptive instruction strategies.

Creating learning progressions is an iterative research process that involves grounding the lower anchor in domains that are accessible to 6th graders, then identifying intermediate levels of understanding through which they pass on their way to attainment of the upper anchor. The learning progression upper anchor is based on experts' views of what quantitative reasoning a scientifically literate citizen should know and be able to do by the 12th grade. While learning progressions in science have incorporated some processes of QR (c.f. Louca, Zacharia, & Constantinou, 2011; Pluta, Chinn, & Duncan, 2011; Schwarz, Reiser, Davis, Kenyon, Acher, Fortus, Shwartz, Hug, & Krajcik, 2009; Stefani & Tsaparlis, 2009; Taylor & Jones, 2009), there presently is no research on a progression examining either the act of quantification or the trajectory of quantitative literacy, quantitative interpretation, and quantitative modeling supporting science as model-building. The development of learning progressions can have far reaching impacts, setting the stage for future research on creating professional development programs for teachers through teaching experiments focused on an interdisciplinary, integrated STEM classroom using place-based and problem-based pedagogies. This in turn initiates the process of curriculum development based on these pedagogies.

The Consortium for Policy Research in Education identified essential elements of learning progressions to be:

- Upper Anchor: target performance or leaning goals which are the end points of learning progression and are defined by societal expectations, analysis of the discipline, and requirements for entry into the next level of education
- Progress Variables: dimensions of understanding, application, and practice that are being developed and tracked over time
- Levels of Achievement: intermediate steps in the developmental pathway(s) traced by a learning progression
- Learning performances: tasks students at a particular level of achievement would be capable of performing
- Assessments: specific measures used to track student development along the hypothesized progression

Learning progressions are based on research in mathematics and science education, cognitive psychology, foundational and generative disciplinary knowledge and practices, and strive for internal conceptual coherence. Our proposed QR frameworks build on these characteristics, incorporating mathematical and statistical frameworks. Learning progression matrices are created by cross tabulating achievement levels (rows in matrix) with progress variables (columns in matrix). The achievement levels are then linked with learning performances that are exemplars drawn from the clinical interviews and written assessments. The learning performances demonstrate student responses at different achievement levels, providing for use of the learning progression as a means to classify student understanding.

A number of learning progressions in science are currently under development including: tracing carbon in ecological systems (Mohan, Chen, & Anderson, 2009), particle model of matter (Merritt, Krajcik, & Swartz, 2008), modeling in science (Schwarz et. al., 2009), genetics (Duncan, Rogat, & Yarden, 2009), chemical reactions (Roseman, Caldwell, Gogos, & Kurth, 2006), data modeling and evolution (Lehrer & Schauble, 2002), explanations and ecology (Songer, Kelcey, & Gotwals, 2009), buoyancy (Kennedy & Wilson, 2006), atomic molecular theory (Smith, Wisner, Anderson, & Krajcik, 2006), and evolution (Cately, Lehrer, & Reiser, 2005). Corcoran, Mosher, and Rogat (2009) provide an excellent overview of three of the above learning progressions. The QR Theme team is informed by these learning progressions as they establish the theoretical foundations and methods for designing quantitative learning progressions. Science progressions that are especially pertinent are those that integrate significant quantitative processes such as the modeling in science, data modeling and evolution, and atomic molecular theory learning progressions. For example the quantitative modeling progression is building on the seminal research done by Schwarz, Reiser, Davis, Kenyon, Acher, Fortus, Shwartz, Hug, and Krajcik (2009) in the MoDeLS project Modeling Designs for Learning Science. Their learning progression for scientific modeling is presented in Figure 2.

The QR Theme team has developed a QR learning progression framework. The framework has yet to be verified by the iterative process which will confirm the lower anchor and intermediate levels of achievement. However the frameworks are based on extensive review of the literature and data from initial interviews with 39 students from grades 6 through 12. The interviews were conducted using three QR Environmental Literacy Assessments developed by the QR Theme team, each based on a key conceptual strand identified by the NSF Pathways science teams: QR Carbon Cycle, QR Water Cycle, and QR Biodiversity. The interviews were 30 to 40 minutes in duration, leading with a question on the macro scale (personal experience of the world, what can be seen with the eye), followed by a question on the landscape scale (global generalizations, what could be seen with a telescope or larger), and finishing with a micro/atomic scale question (hidden mechanisms, what could be seen with a microscope or smaller). The interviews were transcribed and coded, with NVivo employed as a qualitative research tool to identify themes. The QR Theme team also conducted a more indepth analysis of a selected set of the students to study the trajectory of QR development within each strand. For each of the three assessments one student was selected from each of the grades 6 through 12, providing a representative trajectory of students across which QR development was tracked. For example for QR Carbon the trajectory consisted of a sixth, seventh, eighth, tenth, eleventh and twelfth grader. Note that there we no ninth graders in our sample that took the QR Carbon assessment. The results of this analysis, which will be reported in a future research paper, informed our development of the QR learning progression framework.

A conundrum which the QR Theme team has encountered in development of QR learning progression frameworks is separating QR progressions from the mathematics and statistics that underlie the ability to engage in QR. The team began by creating progression frameworks addressing QR processes, including quantitative literacy frameworks for numeracy, proportional reasoning, change, and measure. The team also created frameworks for the multiple representation aspect of quantitative interpretation and an overall framework for quantitative modeling. The problem with this approach is twofold. First, there is the complication of how so many different frameworks with a focus on mathematics and statistics can be integrated into the existing science learning progressions for the Carbon, Water, and Biodiversity Strands. Attempts to cross tabulate the science progressions with the QR frameworks met with little success. In addition in returning to our definition of quantitative reasoning, this is in conflict with the interdisciplinary component of that definition. Second, developing frameworks which focus on mathematical and statistical understandings does not reflect the key aspect of QR as a habit of mind; as the act of using mathematics and statistics in the context of science. While QR requires the use of mathematics and statistics, as we have argued above it is not the same as mathematics and statistics. QR in context is the ability to “see” the mathematics within a context, to choose the appropriate mathematical or statistical tool from a toolbox and apply it within the context, and the ability to move from the mathematical and statistical analysis back to the context to

MoDeLS Learning Progression (Schwarz, Reiser, Davis, Kenyon, Acher, Fortus, Shwartz, Hug, Krajcik, 2009)	
Modeling Science - models as generative tools for predicting and explaining	**Modeling Science - models as changeable entities**
Students construct and use models spontaneously in a range of domains to help their own thinking. Students consider how the world could behave according to various models. Students construct and use models to generate new questions about the behavior or existence of phenomena.	Students consider changes in models to enhance the explanatory power prior to obtaining evidence supporting these changes. Model changes are considered to develop questions that can then be tested against evidence from the phenomena. Students evaluate competing models to consider combining aspects of models that can enhance the explanatory and predictive power.
Students construct and use multiple models to explain and predict more aspects of a group of related phenomena. Students view models as tools that can support their thinking about existing and new phenomena. Students consider alternatives in constructing models based on analyses of the different advantages and weakness for explaining and predicting these alternative models possess.	Students revise models in order to better fit evidence that has been obtained and to improve the articulation of a mechanism in the model. Thus, models are revised to improve their explanatory power. Students compare models to see how different components or relationships fit evidence more completely and provide a more mechanistic explanation of the phenomena.
Students construct and use a model to illustrate and explain how a phenomenon occurs, consistent with the evidence about the phenomenon. Students view models as a means of communicating their understanding of a phenomenon rather than a tool to support their own thinking.	Students revise models based on information from authority (teacher, textbook, peer) rather than evidence gathered from the phenomenon or new explanatory mechanisms. Students make modifications to improve detail, clarity or add new information, without considering how the explanatory power of the model or its fit with empirical evidence is improved.
Students construct and use models that show literal illustrations of a single phenomenon. Students do not view a model as tool to generate new knowledge, but do see models as a means of showing others what the phenomenon looks like.	Students do not expect models to change with new understandings. They talk about models in absolute terms of right or wrong answers. Students compare their models to assess, if they are good or bad replicas of the phenomenon.

Figure 2: MoDeLS Learning Progression

make a decision. Perhaps this view of QR can be captured more in a meta-level quantitative framework like the one we propose in Figure 3, than it can be in a series of detailed frameworks focused on mathematical and statistical tools in the toolbox.

	QR Progress Variable		
Achievement Level	**Quantification**	**Quantitative Interpretation**	**Quantitative Modeling**
Level 4 (Upper Anchor)	**4a** Covariation: comparing, contrasting, relating two or more variables **4b** ability to use quantitative literacy to explore relationships, including the QL aspects of proportional reasoning **4c** QL skills in level 2 extended to include reasoning with operations and patterns **4d** solving ill-defined problems in socio-political contexts using ad-hoc methods (Steen & Madison) **4e** situative view of QR within a community of practice (Shavelson)	**4a** Recognizes and provides quantitative explanations of trends in any model representation, including linear, power, exponential trends **4b** uses covariational concepts of change such as slope (additive change) and multiplicative change **4c** makes predictions using any of the model representations and provides a quantitative account **4d** translates between different models, ability to move from graph to equation and table to equation at least categorically (ie this graph looks exponential) **4e** extend from arithmetic to algebraic, geometric, and statistical methods **4f** recognize limits of arithmetic methods (MAA)	**4a** Collect and organize data in a table, identify trends and make predictions with a quantitative account **4b** create graphic and visual models from data tables, use to determine trends - make predictions **4c** apply best fit to find linear, power, and exponential models, use to determine trends - make predictions **4d** ability to create a model representing a context including science systems models such as box models **4e** extend a given model to answer questions of dynamic change within context **4f** model within context using normal distribution, logarithmic, logistic growth, multivariate, or simulation models **4g** test and refine a model to evaluate scientific evidence and explanations **4h** conduct statistical inference to test hypothesis (Duschl)

	QR Progress Variable		
Achievement Level	**Quantification**	**Quantitative Interpretation**	**Quantitative Modeling**
Level 3	**3a** Variation: ability to manipulate and calculate with a variable to answer questions of change, discover patterns, and draw conclusions **3b** QL aspects of measure, numeracy, descriptive statistics **3c** display confidence with mathematics, cultural appreciation of mathematics, number sense, practical computation skills, interpreting data, and logical thinking (Steen)	**3a** Expand recognition of patterns - trends in graphs and visual representations using change in one variable, recognizing linear vs. curvilinear growth **3b** recognize patterns and trends in tables using change in one variable, may not account for covariance **3c** avoids equation models, understands function as input/output but not as dynamic relationship **3d** interprets science system models qualitatively but avoids quantitative information in model **3e** difficulty with models that embed variable or have more than two interrelated variables **3f** attempts to translate between models if prompted but fails to relate variables between models **3g** makes predictions but provides only qualitative arguments	**3a** Interprets statistical displays, visual representations, and tables with one variable, identifying trends and making predictions with strong quantitative accounts **3b** creates linear models using change but fails to comprehend covariation aspect **3c** extends a given model to account for dynamic change but provides only a qualitative account **3d** draws pictures to represent science systems which are simplistic and lack quantitative detail

	QR Progress Variable		
Achievement Level	**Quantification**	**Quantitative Interpretation**	**Quantitative Modeling**
Level 2	**2a** Act of Quantification: mental construct in which an object within a context is identified, then conceptualized so that the object has attributes that are measurable **2b** object is named creating a variable which is often represented by a letter (Thompson)	**2a** Given a model in any form (table, graph, equation, visual representation, or science system model) the ability to identify variables in the model including axes in a graph, headings in a table, variables in an equation **2b** identify and explain single case (point) in model within the context **2c** recognize increasing/ decreasing trends in graphic models and visual representations (bar graphs, line graphs, histograms) **2d** does not attempt to translate between representations	**2a** Creates visual models to represent single variable data, such as statistical displays (pie charts, histograms) **2b** explores trends and makes predictions without strong quantitative accounts for single variable data **2c** gathers data and organizes it in tables, plots data on two dimensional grid giving scatter plot **2d** fails to explore relationships between data in tables and scatter plots since does not "see" covariation
Level 1 (Lower Anchor)	**1a** QR Avoidance: ignores quantitative data provided in context **1b** resorts to qualitative accounts which are not supported by data	**1a** Fails to relate a given model to the context **1b** avoids using the model, relies on qualitative accounts	**1a** Does not view science as model building and refining so does not attempt to construct models **1b** forced dynamic or low level school science discourse, expect to receive facts and memorize processes

Figure 3: Quantitative Reasoning Learning Progression Framework

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