On Children's Construction of Quantification

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My goal in this paper is to present an analysis of stages in children's construction of discrete quantity and its measurement and how these stages are related to stages in children's construction of continuous quantity and its measurement. Correlating the construction of discrete quantity with the construction of continuous quantity opens a path to emphasizing quantitative reasoning throughout school mathematics. We should not argue the operations that generate an awareness of numerosity [discrete quantity] are necessarily of a different genre than the operations that generate an awareness of length, distance, weight, area, volume, capacity, angles, temperature, time, and speed (Steffe, 2010). As an illustration, consider the following account of Martin, a five-year-old child solving a task involving finding how long a piece of ribbon and a straw would be if they were placed together (Steffe & von Glasersfeld, 1985). Martin had a ruler marked in inches to measure with. Martin first measured the ribbon and found it to be six inches in length and then counted, uttering "1, 2, 3, 4, 5, 6" in synchrony with putting up fingers. He then measured the straw and found it to be twelve inches and again uttered, "1, 2, 3, …, 12" in synchrony with putting up fingers. He then started all over, counting to "twelve" as before and then paused before continuing to utter "13, 14, 15, 16, 17, 18" again in synchrony with putting up fingers and stopped when recognizing a finger pattern for "six." After two or so further similar situations, Martin curtailed counting from one twice, and simply counted the measurement of the second object onto the measurement of the first object rather than count three different times. Martin interpreted "how long" as "how many inches" and seemed to make no distinction between counting and measuring. In fact, he used counting to find "how many inches." Martin used his counting scheme as a measuring scheme and seemed to construe the length units as countable units.

Regardless of the kind of measuring scheme that can be inferred based on children's measuring behavior, the scheme always involves a unit of some kind. In fact, Thompson (1990) defines quantity in such a way that emphasizes the concept of unit.

> A *quantity* is a quality of something that one has conceived as admitting some measurement process. Part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit. (p. 5)

Units and their systems are at the core of any analysis of students' measuring schemes, which, in my analysis is an interpretation of Thompson's "measurement process". It is very easy and in fact commonplace to regard units as a datum in such an analysis; that is, as given things-in-themselves. In contrast, children construct units and their systems that, in turn, constitute basic elements in their construction of measuring schemes. This construction is an elongated process that begins in infanthood the first time a baby recognizes an object that it has experienced before (Piaget, 1937/1954; von Glasersfeld, 1981), and it continues on throughout childhood and beyond (Behr, Harel, Post, & Lesh, 1994; Hackenberg, 2011; Sophian, 2007; Steffe & Olive, 2010; Thompson, 1994).

Discrete Quantitative Measuring Schemes and Their Construction

I have identified four distinct stages in the construction of discrete quantitative measuring schemes that I discuss in this paper. I refer to them as the *perceptual counting scheme*, the *figurative counting scheme*, the *initial number sequence*, and the *explicitly nested number sequence* (Steffe, 2010). These schemes are normally considered as counting schemes, but they qualify as measuring schemes because the "somethings" in Thompson's definition of a quantity are composite wholes of some kind. The qualities of these composite wholes that admit a measurement process [counting] are, in general terms, an awareness of more than one unit item, where the type of unit item is specific to the particular composite whole involved. But rather than speak of qualities, I use "quantitative properties" instead and call the "somethings" that correspond to the four stages, perceptual lots, figurative lots, numerical composites, and composite units. In particular, I call the quantitative property of a perceptual lot an *awareness of perceptual plurality*, which is an awareness of more than one perceptual unit item. In the second stage, I call the quantitative property of a figurative lot an *awareness of figurative plurality*, which is an awareness of more than one figurative unit item. In the case of the numerical composite, the child can form an *image* of a composite of figurative unit items and I call an awareness of the items in the composite an *awareness of figurative numerosity*[1]. At the stage of the explicitly nested number sequence (ENS), an iterable unit of one symbolizes unit items produced by iterating the unit. This symbolic function of an iterable unit of one provides the child with an awareness of more than one unit item in a composite unit of such items without the necessity of making images of them. I call this symbolic awareness an *awareness of arithmetical numerosity.* In what follows, I explain the four stages in the construction of discrete quantitative measuring schemes. A major distinction among the four schemes resides in the distinctions in the type of units that are involved. There are other equally important distinctions, but I initially focus on the distinctions in the type of units. To do so, I begin with the unitizing operation, the mental operation that produces units of all kinds.

The Unitizing Operation

The *unitizing operation*—making a unit--is the operation of the mind that produces units and it is present from birth onwards (von Glasersfeld, 1981). It functions recursively and, by operating on its own products, eventually produces numerical and quantitative units (Steffe, von Glasersfeld, Richards, & Cobb, 1983). I have always been surprised that von Glasersfeld's (1981) attentional model has not been widely used in mathematics education research because it provides a way of thinking about the unitizing operation that is involved in the construction of object concepts by infants and, from that time forward, systems of units. It does not specify the material of construction, but it does provide a model of the operation that produces unitary items of all kinds.

Beginning with the concept of the manifold, von Glasersfeld provided an account of experience.

> The "manifold", then, is the raw material, the stuff on which constructive perception and reason can operate. … In present-day neurophysiology one would say, it is the totality of electrochemical impulses continuously generated by the sensory organs of the system. Even if one assumes that these impulses are caused by differences of an ontic substrate they cannot convey qualitative information, because qualitatively they

1 A numerosity can be formed by repeatedly instantiating an arithmetical unit in an experiential situation synchronous with uttering number words.

> are all the same. *Experience, thus, is what the subject coordinates (constructs) out of elements of the manifold* [Italics added]. (von Glasersfeld, 1995, pp. 40-41)

The attentional model provides an account of the coordinating mechanism. Attention is not understood as a conscious state that can be extended through periods of time.

> Instead, I intend a pulse-like succession of moments of attention, each one of which may or may not be "focused" on some neural event in the organism. By "focused" I intend no more than that an attentional pulse is made to coincide with some other signal (from the multitude that more or less continuously pervades the organism's nervous system) and thus *allows it to be registered* [Italics added]. An "unfocused" pulse is one that registers no content" (von Glasersfeld, 1981, p. 85).

In his paper, Glasersfeld provides a convincing argument *that attention operates above and independently of sensation* and so functions as an organizing principle. Two salient examples he gave are shifting our attention at will from one part of the visual field to another without moving our eyes and deliberately focusing on any one of several simultaneous auditory signal sequences (the cocktail party effect). So, if attention can shift from one place to another in the experiential field, it must have a means of regarding (focusing on) those places and disregarding ("unfocused pulse") whatever lies in between.

From our adult perspective, we can cut things out of experiential backgrounds and regard the things as unitary items. We can create unitary items in the absence of sensory differences as well as in their presence, such as separating a blank sheet of paper into two parts or willfully mark a cloudless sky into parts (the intensity of the experience of a glacier to me is overwhelming and I am not able to cut foreground out of background). Attention provides a way of thinking about how the arithmetical unit, one, is constructed as an abstract entity having no necessary material meaning, how the number seven is constructed as an entity containing other abstract entities, how a point is constructed as having no sensory content, how a line is constructed as having extension but no sensory content, and a host of other mental constructs that we consider as mathematical concepts. It also provides an account of taking an experience of a moving object as an entity and "holding it still" or "letting it go", both of which are essential in establishing a rate concept.

Prior to the time of recognition, von Glasersfeld's (1981) model provides insight into the nature of the sensory-motor items that babies establish. A group of co-occurring sensory-motor signals becomes a "whole" or "object" when an unbroken sequence of attentional pulses is focused on these signals and the sequence is framed or bounded by an unfocused pulse at both ends. The unfocused pulses provide closure and set the sequence of contiguous focused pulses apart from prior and subsequent attentional pulses. A focused moment of attention registers sensory material and an unfocused moment of attention can be regarded as a blank space. The records of making a sensory-motor item, or an item of experience, were graphically illustrated in terms of an *attentional pattern* as shown in Figure I (1981, p. 87).

$$\mathrm{O}\ \mathrm{I}_a\ \mathrm{I}_b \ldots \mathrm{I}_k\ \mathrm{O}$$

Figure 1: An attentional pattern: Sensory-Motor item

The unfocused moments of attention are designated by "O" and bound the focused moments of attention designated by "I". The letters a, b, . . . , k designate sensory material selected by attention and this sensory material is registered as records of experience. I emphasize that the *attentional pattern* is established as a result of individual-environment interaction and the process it symbolizes constitutes a model of the operation that is involved in compounding sensory signals together in the immediate here-and-now to form items of experience—*the unitizing operation*.

> The hypothesis I am here proposing is that these unitizing operations consist in the differential distribution of focused and unfocused attentional pulses. A group of co-occurring sensory-motor signals becomes a "whole" or "thing" or "object" when an unbroken sequence of attentional pulses is focused on these signals and the sequence is framed or bounded by an unfocused pulse at both ends. The unfocused pulses provide closure and set the sequence of contiguous focused pulses apart from prior and subsequent attentional pulses (von Glasersfeld, 1981, p. 89).

As essential as establishing sensory items might be in the immediate here and now, if that were the only way the unitizing operation was used, we would always live in the moment and have no recollection of our past experiences. However, in the process of assembling a current sensory item, one or more sensory signals might trigger an activation of a previous attentional pattern. In that case, the infant might use the previous pattern to re-focus attention on the current sensory item. This re-focusing of attention constitutes recognition of the current item using a prior attentional pattern and marks the beginning of using the unitizing operation recursively to re-process a current sensory-motor item. In reprocessing, novel records may destroy prior records, prior records may be re-recorded or not recorded at all, and novel sensory material may be recorded. In any event, further assimilation using the recognition pattern involves selection and variation and opens the possibility that it will be modified in its use[2]. Realizing that selection and variation can be involved in recognition highlights the importance of repeated experience in the construction of object concepts (Cooper, 1991).

Awareness of Perceptual Plurality: A Quantitative Property of a Perceptual Lot

There are good reasons why I have spent time in this paper developing the concept of the unitizing operation. The first is to emphasize its use in constructing experiential items like balls, flowers, words, etc. The second is to stress that these constructed items are used in the construction of number and quantity. The third is to stress that using the unitizing operation recursively is an indicator of the beginnings of generating an image of the item, which Glasersfeld (1991) refers to as re-presenting the item. Re-presenting an item is the minimal indicator that an infant has internalized the item and, thus, has constructed an object concept. But it is not sufficient because Piaget's (1937/1954) studies on object permanence indicate that if the emergence of such awareness in visualized imagination is coordinated with an awareness of its *location* in immediate experience, the child is said to have constructed an *externalized* permanent object. Such externalized permanent objects in their continual construction are the objects that comprise our immediate experiential reality.

Externalized, permanent objects are necessary in students' construction of *perceptual collections*. Let us say, for example, that a child recognizes a perceptual situation an observer would associate with the word "cup." The child may continue to explore its visual field and assimilate another combination of sensory signals that an observer would regard as a different cup. The child could experience a recurrence in the second recognition episode if a current sensory feature triggers a re-presentation of the preceding recognition episode. If the child continued on exploring its visual field and established yet another combination of sensory signals that an

2 When an attentional pattern is used in recognition, I refer to it as a recognition template.

observer would regard as yet a different cup, the child might experience the sensory feature as again recurrent. If the recurrent sensory features that are involved are permanently recorded in an attentional pattern, this constitutes what I regard as an empirical abstraction[3].

The abstraction opens the pathway to using the abstracted attentional pattern to focus attention on the recurrent sensory features of each sensory item, or to reprocess the sensory items. Reprocessing forms the operational basis of categorizing the items together into a *collection of perceptual items*. This process of empirical abstraction produces a composite whole, but it is only experientially bounded in the perceptual field of the child. The recognition template that contains the common sensory features that the child has abstracted fits von Glasersfeld's (1982) explanation of a concept[4]. Even though the child may be quite able to recognize a cup or to repeatedly point to a quadruplet of cups while saying, "cup, cup, cup, cup", using the concept repeatedly in re-presentation to produce the quadruplet in visualized imagination may not be possible for the child even though the child can use it in visualizing a singular cup image. In that case, like a singular sensory item prior to the construction of externalized permanent objects, the perceptual collection is only momentarily established and is yet to be constructed as a *permanent object,* which is a crucial step in the construction of number.

Nevertheless, reprocessing perceptual items as a means of taking them together opens the way for another abstraction that I regard as a pseudo-empirical abstraction[5]. For example, if a child establishes an abstracted template as I have described above and uses it in re-processing the sensory items used in abstracting the template, and then, perhaps fortuitously, encounters an item that is not recognized using the abstracted template [from the observer's perspective, this would be a heterogeneous item], this encourages focusing attention on the *unitary wholeness* of each perceptual item rather than on the particular sensory material that is recorded in the abstracted template. Focusing on the unitary wholeness of the sensory items is *an operation of unitizing the perceptual items* that produces *perceptual unit items*. The attentional pattern of a perceptual unit item is diagrammed in Figure 2.

$$0\ \underset{n}{I}\ 0$$

Figure 2: The attentional pattern of a perceptual unit item

The notation in Figure 2 is used to designate a single attentional moment focused on the unitariness of a sensory-motor item. In this, "n" is used to denote the necessity of having some, but no particular, sensory-motor material on which to focus.

This development of perceptual unit items opens the possibility of the child categorizing non-homogeneous items together on the basis of their unitariness—"things" that go together *because they are put together*, and it fits with how Inhelder & Piaget (1964) in part described non-graphic collections:

3 This interpretation provides an operational basis for Piaget's (1966, p. 188-89) empirical abstraction.

4 The term "concept" "refers to any structure that has been abstracted from the process of experiential construction as recurrently usable, for instance, for the purpose of relating or classifying experiential situations. To be called "concept" these constructs must be stable enough to be re-presented in the absence of perceptual "input" (von Glasersfeld, 1982, p. 194)

5 Pseudo-empirical abstraction is like empirical abstraction, but the perceived properties are actually introduced into these objects by the subject's activities (von Glasersfeld, 1991).

> However, he [Eli 5; 6] then goes on to make collections based on similarity alone: the fish with the birds, etc, "because they're all animals", then the people, then pots, etc. "because they're all things for making supper" (p. 56).

Once established, if a perceptual unit item is used to reprocess a collection of perceptual items, this produces a collection of perceptual *unit* items. Records of the reprocessing action that categorizes perceptual unit items together are contained in an attentional pattern as diagrammed in Figure 3. I call this pattern a perceptual LOT—a collection of perceptual unit items--and use parentheses to denote that the action of categorizing occurred within experiential boundaries.

$$(OIO_n\ OIO_n\ OIO_n\ OIO_n\ \ldots\ OIO_n)$$

Figure 3: The attentional pattern of a perceptual lot

When a perceptual LOT is activated in an experiential situation, the unit items of the pattern are activated, which provides the child with an awareness of more than one perceptual unit item. I refer to this awareness as an *awareness of perceptual plurality*[6]. In a perceptual LOT, the child senses the co-occurrence of more than one perceptual unit item in immediate experience. Such an awareness of perceptual plurality can produce a sense of indefiniteness, or a lack of closure, which in turn can serve as a goal that activates an activity of counting if the activity is available to the child. The child counts for the purpose of making definite the sense of indefiniteness.

I regard an awareness of perceptual plurality as a "quantitative"[7] property of a perceptual LOT. It is what permits children who are restricted to establishing the perceptual unit items of a perceptual LOT as countable items to engage in purposeful counting activity in that it is their goal to make definite their sense of indefiniteness induced by their awareness of more than one perceptual unit item. However, if the LOT is hidden and, if a child is unable to count the items of the LOT, the child is called a counter of perceptual unit items. They know how to count, but they need perceptual unit items in their visual field in order to carry out the activity. In fact, I constructed the concept of a perceptual LOT to explain such counting behavior of some six-year-old children (Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe, 1994). I refer to such children as being in the perceptual stage in the construction of their counting scheme. *For these children, perceptual LOTS are yet to be constructed as externalized, permanent objects*.

Awareness of Figurative Plurality: A Quantitative Property of a Figurative Lot

Although a *collection* of perceptual unit items may yet to be constructed as an externalized, permanent object by a child, the involved individual perceptual unit items are indeed permanent objects. What this means is that such children can remain aware of an experience of a collection of perceptual unit items when the collection is not within their range of perception or action. For example, if such a child puts seven tiles beneath a cloth, the child would be aware that tiles were hidden because the child can produce a visualized image of a tile[8]. But, what operations are involved in a child producing an image of a collection of perceptual unit items? Toward explaining those operations, I offer the following exchange with Jason from an interview that

6 I use "perceptual plurality" to refer to more than one perceptual unit item within experiential boundaries. My choice of "plurality" rather than "collection" is made to accentuate the child's awareness of more than one perceptual unit item rather than an awareness of an unitary whole containing the unit items.

7 Quotation marks are used to indicate that we are speaking of quantity on a perceptual level.

8 An inability to create an image of a perceptual unit item would exclude experiencing perceptual unit items in their immediate absence, for then the child could only recognize the item in its immediate presence.

occurred on October 16th of his first grade in school (Steffe, 1994, p. 142). Jason had not established finger patterns as meanings of number words at this time in the teaching experiment except for “two” and “ten.” In fact, it was not until December 3rd of the same school year that I observed Jason use a finger pattern for “three”.

Protocol I: Constructing a figurative collection.

T: (Presents two cloths that Jason takes as covering squares). There are six here and five here. How many squares are there altogether?

J: (Sequentially moves fingers on his left hand, moving his index finger twice) 1-2-3-4-5-6. (Continues sequentially moving fingers on his left hand) 7-8 (middle and ring finger) 9-10 (index and middle finger) 11-12 (moving no fingers).

I can see no reason to believe that Jason needed a numerical concept of “six” in order to count as he did. I do believe that “six” referred to an image of tiles, but there was no reason to believe that six individual tiles appeared to co-occur in some pattern for Jason. His counting meaning of “six” alleviated that necessity—for Jason, “six” meant to count six tiles in his visual field. The crucial factor in Jason’s being able to sequentially produce images of hidden tiles, however minimal, was that he could use his object concept, tile, to sequentially produce more than one visualized image of a tile. Jason still needed sensory items to count [moving his fingers], and the sensory items were countable substitutes for the tiles he visualized in each counting act. But Jason was “in” the visualizing activity, not “outside” of it. That is, Jason was yet to hold the visualized images of tile at a distance and reflect on them.

A child repeatedly producing more than one visualized image of a perceptual unit item [a figurative item] is a significant event in a sequence of events that leads to the construction of a unit of arithmetical units. I assume that in producing a sequence of figurative items, the child uses the items of the involved perceptual LOT structure in re-presentation in generating the sequence because, without them, there would be nothing to propel the process forward[9]. If, in generating the sequence of figurative items, the child experiences a recurrence in the production of the figurative items, this opens the possibility of the figurative items being registered or recorded in a new attentional pattern that I consider as a *figurative unit item*. This is a re-capitulation of the account of empirical abstraction but this time on the figurative level. The figurative unit item is illustrated in Figure 4. The broken line indicates records of figurative material that was recorded in the attentional pattern.

O I O

Figure 4: The attentional pattern of a figurative unit item.

A figurative unit item is not as abstract as Piaget’s arithmetical unit because the child is still dealing with items of its ordinary experience and not with the more abstracted units of number. Its construction opens the possibility of using it sequentially to produce visualized images of perceptual unit items, the result of which I call a figurative LOT structure diagramed in Figure 5. The dotted lines indicate records of figurative material and the parentheses indicate that a figurative LOT is experientially bounded by the beginning and end of the process that produces it.

9 I also assume that the production of a sequence of figurative items occurs in a context such as a child imagining the toys in a toy box.

(O I O O I O O I O … O I O)

Figure 5: The attentional structure of a figurative LOT

When the attentional structure of a figurative LOT is activated in an experiential situation, such as covered tiles, the unit items of the structure are activated and provide the child with an awareness of more than one figurative unit item, which I refer to as an *awareness of figurative plurality*. In this, the child has a sense of the co-occurrence of more than one figurative unit item in immediate experience. This awareness of figurative plurality can also produce a sense of indefiniteness, or a lack of closure, which, in turn, can serve as a goal that activates the activity of a counting scheme. I regard an awareness of figurative plurality as a quantitative property of a figurative LOT introduced by the activity of the child.

The construction of figurative LOTS provides an account of the reconstitution of perceptual LOTS *as permanent objects* similar to how the construction of externalized permanent object concepts provided an account of the reconstitution of recognizable sensory items[10]. But there is a restriction in the production of figurative LOTS in that they are not used to produce visualized images of more than one figurative item unless there is a perceptual situation that triggers the operations that produced the LOT. The abstraction that produced them is compatible with Piaget's pseudo-empirical abstraction.

The construction of figurative LOTS is a crucial step in the construction of number. I constructed the concept to explain six-year-old children who were able to count items that were hidden from view but who also always started counting from "one" such as the case of Jason (Steffe, 1994). Jason's motor acts of putting up fingers stood in for the elements of the LOT structure and his counting scheme was judged to be figurative rather than numerical.

The inferred concepts of a permanent object concept, a perceptual unit item, a perceptual LOT, a figurative unit item, and a figurative LOT all refer to pre-numerical structures that are produced by *coordinating the unitizing operation and the operation of re-presentation*. A figurative LOT is an internalized composite whole that is analogous to the construction of object concepts, but it is not yet an externalized object concept. The operations that produce a figurative LOT are still pre-numerical operations[11].

Numerical Composites and Composite Units

I now turn to a child with whom I worked, Tyrone, who was six years of age and had already established figurative patterns for the number words "two" through "four", but had no figurative pattern for "five" to keep track of counting. On the 15th of October of his 1st Grade year, the child was presented with two hidden collections of counters, one of seven and one of five, and was asked to find how many counters were hidden. He counted over the screen hiding the seven counters "one, two, three, four, five, six, seven" while pointing to seven places on the screen, which indicates that he could at least sequentially re-present the counters. He then continued counting over the screen hiding five counters as well, uttering "eight, nine, ten, eleven, twelve, thirteen" synchronous with pointing to places on the screen that he took as hiding the counters. He realized that he didn't know when to stop counting [he had no pattern for "five" that he could use to keep track] and independently started counting from "one" again, this time monitoring the

10 The initial construction of object concepts is at the internalized level.

11 The distinction between counters of figurative items and counters of motor unit items constitute the distinctions in the behavioral indicators of children who sequentially re-present the items of a perceptual Lot in counting and children who re-present a figurative Lot prior to counting. In Protocol I, I assume the latter. (cf. Steffe, von Glasersfeld, Richards, & Cobb, 1983).

continuation of counting activity beyond "seven" (Steffe & Cobb, 1988, p. 153).

Monitoring counting. When Tyrone independently started over again and monitored counting, it was a crucial indicator of the operations that create numbers. Although self-regulation might seem to be too stringent of a behavioral criterion to infer numerical operating, in my work with children I have found it to be a critical indicator of children's construction of numerical operations (Steffe, 1994). Self-regulation also indicates that Tyrone wanted to count five more times beyond counting to seven and that he knew that he had not counted that many times. When he recounted over the screen hiding five counters, he deliberately touched the screen hiding the five counters while looking intently at his points of contact and stopped when he reached "twelve"[12]. So, I inferred that he monitored his counting acts and accounted for monitoring via re-presenting his figurative counting acts "eight, nine" and applying the unitizing operation to these re-presented counting acts (Steffe & Cobb, 1988, pp 308ff). This is an act of abstraction and creates a pair of arithmetical unit items [two attentional patterns] containing records of the figurative counting acts from which they were abstracted. I refer to these abstracted counting acts as *interiorized* as opposed to the re-presented counting acts, which were still only *internalized*, figurative unit items.

Continuing on in this way, Tyrone created a sequence of interiorized counting acts [arithmetical unit items], "eight, nine, ten, eleven, twelve," which constituted a numerical pattern for five. The sequence of *arithmetical unit items* that Tyrone created is illustrated in Figure 6 as an attentional pattern whose elements are arithmetical unit items, or "slots" that can be filled with sensory material of any kind.

$$(0\underline{\,I\,}0\;\;0\underline{\,I\,}0\;\;0\underline{\,I\,}0\;\;0\underline{\,I\,}0\;\;0\underline{\,I\,}0)$$
$$8\quad\;\; 9\quad\;\; 10\quad\; 11\quad\; 12$$

Figure 6: An arithmetical pattern for "five".

The arithmetical pattern for "five" was only experientially bounded, not attentionally bounded, which it would be if Tyrone had focused his attention on the pattern as an entity. However, he was in the process of constructing the numerical pattern rather than using it in further operating, so constructing the pattern as a composite unit in that case would be spectacular indeed. It is useful to think of a numerical pattern as a composite of co-occurring arithmetical unit items[13].

I indicated that Tyrone monitored counting, and it is possible to be more explicit about what I mean. When Tyrone anticipated counting beyond the first seven counting acts, it would be necessary for him to create countable items or he wouldn't have continued counting. But there were no items in his visual field, so he created countable items in re-presentation, which is to say that he produced more than one figurative unit item as the elements he intended to count[14]. The arithmetical units that he created as he counted, say "eight, nine", were counted items and, as such, they were a part of the sequence of counted items starting from "one". But they were also a part of the countable items. When he counted to "seven", it wasn't necessary for a counting act of the sequence to have this dual function because he was enacting his meaning for "seven", which was to actually count from "one" through "seven". When he said, "five", for example, and pointed at the cloth, it wouldn't be necessary for there to be an intentional separation between the items he had counted up to that point and the items yet to be counted. He

12 Monitoring counting is a strong behavioral indicator of the construction of a figurative LOT.

13 A numerical pattern can also consist of a composite of abstract unit items; that is, interiorized figurative unit items that do not contain records of counting.

14 His goal to make definite what was indefinite drove his continuation of counting.

was simply in the midst of counting to "seven", which was, of course, made possible because his counting acts were substitutes for the hidden counters. But, when he said, "eight, nine", it was necessary for him to constitute the pair of counted items as "two", which would in that case be a part of the items he intended to count[15]. In other words, the results of counting, which were arithmetical unit items, became a part of the countable items if for no other reason than after saying, for example, "eight, nine", he went on to count the next countable item, "ten". So, the arithmetical items "eight, nine" were part of the emerging elements of his numerical pattern for "five" that he was creating. Because of the feedback system that he was creating, his counting scheme was being constructed as a self-referencing scheme. He was indeed constructing his counting scheme as a recursive scheme, which means that he could take records of counting acts as countable items.

I also regard reprocessing counting acts as did Tyrone as a *disembbeding operation* that interiorizes segments of counting acts ["9, 10, 11, 12"] and transforms them into counting operations. If the numerical pattern that Tyrone established was permanent; i.e., if the arithmetical units did not decay, then I would say that he had constructed an *externalized* permanent object concept [a numerical pattern] for "five[16]". I also consider monitoring counting as an accommodation in his figurative counting scheme[17] because Tyrone was observed monitoring counting in other counting episodes a month later (Steffe & Cobb, 1988, p. 154 ff). However, a major reorganization of his counting scheme had occurred at the time of the second observation of monitoring counting in that he became able to count-on as well. His number words now referred to numerical composites as in the example of counting-on that I explained above, and he no longer had to count from one to give meaning to number words.

Metamorphosis of the counting scheme. The "jump" or "discontinuity" in the constructive process involved a metamorphosis of Tyrone's figurative counting scheme that I explained in the following way. The abstracted numerical patterns induced a systemic disequilibrium in Tyrone's figurative counting scheme in that the counting acts from eight through twelve were interiorized and the others were still only internalized. They were at two different levels and this was the systemic perturbation that drove monitoring counting until equilibrium was restored. This auto-regulated process produces what I have called the initial number sequence, which is a sequence of arithmetical unit items containing records of counting acts. Its attentional structure is diagrammed in Figure 7. It is important to note that the INS is yet to be taken as a unit; it is a sequence of arithmetical units but not a unit containing the sequence.

$$_{1}(0\underset{2}{\underline{\mathrm{I}}}0\ \ 0\underset{3}{\underline{\mathrm{I}}}0\ \ 0\underset{4}{\underline{\mathrm{I}}}0\ \ 0\underline{\mathrm{I}}0_{n}\ldots\ \ 0\,\mathrm{I}\,0)$$

Figure 7: The attentional structure of an initial number sequence.

A number word of the initial number sequence, like "seven", can activate a sequence of *arithmetical unit items* of specific numerosity. I consider such a sequence as an elaborated numerical pattern in the sense that the child is aware of the elements of the pattern but not the pattern as one thing—as an entity. Moreover, the child may be aware of elements at the beginning and the

15 I do not believe that he actually said "two" to himself or any of the other number words up to "five". At the most, there was a nonverbal awareness of the numerosity of the counted items, which might be thought of as "not five".

16 Being "externalized" means that Tyrone could willfully produce images of the arithmetical units in the absence of any sensory material in his perceptual field as well as use it as a recognition template.

17 An accommodation is an operation that produces a change, not the change.

end of the pattern, but not necessarily of all of the elements in between. When activated, the numerical composite in turn activates the production of figurative material that refers to counting. An activated numerical composite is somewhat like a resonating tuning fork with the stipulation that it's resonating creates an image of counting. The image might be simply some minimal re-presentation of the involved number words that symbolize counting, or it might be a plurality of flecks that symbolize countable perceptual items. The hypothetical *slots* [elements of the attentional pattern] that contain records of experience that are used in producing operative images are realized experientially only upon activation of the program of operations that produce them, and they are realized only as images produced by means of the records contained in them.

Iterable unit items and composite units. The construction of the INS involves reprocessing actual counting acts and, thereby, it can be appropriately thought of as a sequence of interiorized counting acts. A number word of the INS symbolizes the initial segment of the number sequence from "one" up to and including the given number word, which is a numerical concept. But it does not symbolize a composite unit containing that initial segment. In December of Tyrone's 2nd Grade year, Tyrone operated in such a way that I inferred that he constructed ten as a composite unit in operating (Steffe & Cobb, 1988, pp. 162-63).

Protocol II: Tyrone constructing ten as a composite unit.

T: (Places three strips[18] in front of Tyrone and hides two strips and two more squares) How many are there here (the three strips that are visible)?

Ty: Thirty.

T: If I put these (hidden strips and squares) with them, there would be fifty-two. How many would be under here? How many little squares and how many strips?

Ty: (Sequentially puts up twenty-two fingers in synchrony with subvocal utterances. He then says, after a crucial pause where he appeared to be in deep thought): There would be two of these (slaps the strips) and two of them (gesturing toward a pile of squares).

T: How did you figure that out so well?

Ty: I counted, 10-20-30 (I synchrony with moving strips); and then 40-50; and then 51-52.

When Tyrone put up twenty-two fingers in synchrony with subvocal utterances, I inferred that he segmented counting into two unitary parts [two numerical composites of ten] and two units of one because he said that there would be two of each. The pause where he appeared to be in deep thought, when coupled with his re-enactment of counting by ten and one in the last two lines of the protocol, is a solid indication that he reviewed the twenty-two counting acts that he established when counting by one and established two composite units of ten counting acts and then two more counting acts.

To *review the twenty-two counting acts* means to re-present the result of producing them and then, using the unitizing operation, "scan" the re-presented result and recursively unitize the scanned result[19]. In this process, I assume that his finger pattern for ten played an incontestable functional role. In the scan, the unitizing operation would need to be applied only to a sufficient number of re-presented counting acts, sufficient in the sense that ten counting acts would be in-

18 There were ten squares in each strip.

19 "Recursive" is used to indicate that the unitizing operation is applied to previous results of using it. I use "uniting" to distinguish unitizing a composite of counting acts from unitizing a single counting act.

dicated by however many were re-presented. All that would be needed would be experientially bounded remnants, such as re-presented counted fingers indicating a few counting acts. Tyrone could then take the experientially bounded remnants as one thing using the unitizing operation and produce an attentionally bounded numerical composite diagramed in Figure 8. The notation "010" and the numerals under the notation refer to the result of applying the unitizing operation to a represented counting act and the zero's at each end indicate the result of applying the unitizing operation to the results of the scan. The construction of the composite unit of ten counting acts is an act of reflective abstraction and, for Tyrone, it heralded the construction of the explicitly nested number sequence [ENS][20].

0(010 010 010 010 010 010 010 010 010 010)0
31 32 33 34 35 36 37 38 39 40

Figure 8: An attentional structure of a composite unit of ten counting acts.

In a teaching episode held on the 9th of March of his 2nd Grade year, Tyrone counted by ten and by one from the minuend to the difference to find the missing subtrahend in the equation 71 - __ = 39 (Steffe & Cobb, 1988, p. 167).

Protocol III: Solving a missing minuend problem.

T: (Places the sentence "71 - __ = 39" in front of Tyrone) We have seventy-one take away a number and that leaves us with thirty-nine.

Ty: (Sequentially puts up three fingers on his left hand) 61-51-41. (He then puts up a finger on his right hand and pauses) 41—40!—39. (He then places "32" in the blank space".

T: (Removes "32" from the blank space) Let's pretend we don't know the answer to that one. Could you find the answer by counting forward?

Ty: (Sequentially puts up three fingers on his right hand) 49-59-69—69—69. (Shifts to his left hand and sequentially puts up three fingers) 70-71-72 (looks into space as if he is aware of a mistake).

T: (Intervenes, suggesting that Tyrone has "71", not "72". Tyrone goes on to work out the mistake.

Tyrone counting from 71 down to 39 indicated that, for him, 39 was a composite unit comprising the first 39 individual units of 71. It also indicated an explicit awareness that 39 and its remainder in 71were composite units composing 71, and that these composite units were disembedded from their inclusions as two numbers separated from 71. That is, to assimilate the missing subtrahend problem using his counting-down-to scheme, 39 would have to be included in 71. But, this inclusion serves as the basis for "71 – 39 = __" [71 take away 39]. Subtraction in the missing subtrahend problem is *one of difference*; that is, "How far is it from 71 down to 39?", which can translate into "How far is it from 39 up to 71?", a possibility that the teacher investigated.

Although there was no reason to infer that counting-down-to by tens and ones and counting-up-to by tens and ones had been constructed as reversible schemes, there is good reason to

20 One can think about the attentional structure as an interiorized counted finger pattern consisting of two open hands.

infer that Tyrone has constructed the *disembedding operation*. There is also solid indication that Tyrone regarded a *composite unit of ten as an entity*. To regard ten as an entity implies that he used ten as if it were a unit of one while maintaining a distinction between the entity that comprised his unit of ten and the entities that comprised his units of one. That is, he regarded ten as a singleton unit and, to do so, implies that *one was an iterable unit*. That is, rather than contain ten individual unit items, his unit of ten contained a unit of one that could be iterated ten times to "fill out" his unit of ten.

That his unit of one was an iterable unit was corroborated by the way he operated in another task to find how many bags of ten blocks per bag were hidden under a cloth, after he was told that another cloth covered 39 little blocks and that 89 little blocks were hidden under both cloths. Tyrone counted by ten and said "five". He then volunteered that there were fifty little blocks under the cloth as well, and justified his answer by explaining, "There are five bags and if you took them out of the bags, there would be fifty" (Steffe & Cobb, 1988, p. 170). The rapidity with which Tyrone switched from units of ten to units of one contraindicates that Tyrone conjured up an image of ten individual unit items when counting by ten and when he switched from five bags to fifty blocks. This contraindication is important for inferring that his unit of one was an iterable unit because an iterable unit implies the elements of the composite unit from which it was abstracted.

Further, I infer that, prior to counting by tens and ones in the missing subtrahend task, Tyrone established an image of a unit of ten however minimal that symbolized how many units of ten it was from 71 down to 39[21]. That is, I infer that Tyrone had established his unit of ten as an iterable unit. To be constituted as an iterable unit regardless of whether the unit is a composite unit or a singleton unit of one, an image of the unit must refer to a possible image of a bounded plurality of identical units from which the iterable unit was abstracted. But, when the unit is implemented in experience, rather than each implemented unit being a different unit of ten than those preceding, each is construed as identical to the previously implemented units.

There are three principal operations of the ENS with respect to units of one. The first is that one has been constituted as an iterable unit. In this case, 71, for example, can be conceived of as a unit of one iterated 71 times. The child can still regard 71 as a unit containing 71 individual units of one, so the composite quality of 71 is maintained although it is regarded as a unitary item—as an entity. This provides children with great economy in operating that was demonstrated by Tyrone's solution to the missing subtrahend problem. As a singular entity, 39 was regarded by Tyrone as disembedded from 71 while also the first part of 71 where the other part was the difference of 71 and 39; that is, the remainder of 39 in 71. By suppressing the composite quality of both 71 and 39, the composite quality of the difference—the unknown numerosity of the difference—could become the focus of attention and the goal could be to make the unknown numerosity known by counting from 71 down to 39.[22, 23] Although I didn't make it explicit in the discussion of Tyrone's ways and means of operating, Tyrone *took the number sequence symbolized by the numeral 71 as its own input*. That is, Tyrone's countable items were elements of his number sequence. I refer to this way of operating as *the recursive property* of the ENS.

21 I also infer that he established an image of a unit of one that he would use to complete counting after he exhausted counting by ten.

22 Of course, Tyrone did more in that his goal was to make the unknown numerosity known by counting down to by tens and ones, which was made possible by his iterable unit of ten.

23 Tyrone constructed ten as an iterable unit as a result of working with us in teaching episodes. Iterable units are not something that comes with the construction of the ENS, but they can be abstracted from activity at the ENS level. Further, the construction of ten as an iterable unit does not imply that other composite units are constructed as iterable.

The three properties of the ENS are exemplified by a child named Johanna when she reasoned strategically. Johanna was asked to take twelve blocks. The interviewer took some more and told Johanna that together they had nineteen and asked how many he took. After sitting silently for about 20 seconds, Johanna said "seven" and explained, "Well, ten plus nine is nineteen; and I take away two – I mean, ten plus two is twelve, and nine take away two is seven" (Steffe 1992, p. 291). Johanna disembedded ten and nine from nineteen and then operated on the two numbers until she transformed them into twelve and another number that, when added to twelve, would make nineteen.

The strategic reasoning exemplified by Johanna is made possible by the ability of an ENS child to form an image of nineteen as a singular unit item that can be iterated by counting nineteen times to fill out a composite unit containing nineteen items. The child can "go" to the implication of its operative image and generate an image of component parts of nineteen and *operate on these images*. I highlight the ability of the child to disembed an image of the continuation of ten in nineteen and to unite the items of this image into a composite unit without destroying the composite unit structure that contains it. Iterating a unit item, disembedding a numerical part from a numerical whole, and the recursive property, the three principal operations of the ENS, permit taking parts or wholes as material of further operating as did Johanna in strategic reasoning. So, I do not regard a composite unit structure as an object the child has stored somewhere in memory. Rather, I regard it as a product of an ensemble of possible operations that is symbolized by number words. In regarding composite unit structures in this way, I am in agreement with Dörfler (1996) about not pursuing a theory of mind where the mind has or contains mental objects corresponding to numbers, natural or rational. I highlight the ability of the child to disembed an image of the continuation of ten in nineteen and to unite the items of this image into a composite unit without destroying the composite unit structure that contains it.

Characteristics of a Genuine Discrete Quantitative Measuring Scheme

I have outlined steps in the children's construction of the ENS, which I interpret as the first *genuine* discrete quantitative measuring scheme. There are several characteristics of a discrete quantitative scheme that qualifies it as a genuine measuring scheme. The first of these is that the measuring unit has been constructed as an iterable unit. This implies that the "object" to be measured ["thing" in Thompson's terms] is conceived of by the child as a composite unit, which is the second characteristic. The third characteristic is that the quantity ["quality" in Thompson's terms] is an indefinite numerosity of a composite unit. This third characteristic is exemplified by Johanna when she reasons strategically to find how many blocks would need to be added to her twelve to make nineteen blocks [the indefinite numerosity]. Not only do I infer that Johanna constructed an indefinite numerosity, I infer also that her sense of an indefinite numerosity constituted her goal that drove her further reasoning, which is the fourth characteristic.

The disembedding operation and the recursive property of the ENS, the fifth and sixth characteristics, may not at first glance seem to be necessary characteristics of a genuine discrete quantitative measuring scheme. But the disembedding operation is already implicated in the construction of the INS, so the operation is a constitutive aspect of the ENS. An example of its importance is readily demonstrated in Tyrone's construction that there would be two strips of ten squares and two individual squares hidden beneath a cloth when there were three strips visible and fifty-two squares total [Protocol II]. The construction of ten as a measurement unit involves disembedding units of ten sequential counted items from a sequence of counted items, where the countable items are units of one. Taking the number sequence as its own input—recursive reasoning--is inextricably intertwined with disembedding segments of the number sequence from itself, so it is necessary to include recursive reasoning as an essential characteristic of a discrete quantity measuring scheme. Rather than attempt to teach ten as a unit in the context of teaching place value, the construction of ten as an iterable unit coordinated with one as an iterable unit is an important step in the construction of a more general discrete quantitative measuring scheme involving the units, ten and one, on which the construction of the base ten numeration system should be based.

Itineraries in Children's Construction of Discrete Quantitative Measuring Schemes

The perceptual counting scheme. Professor Bob Wright of Southern Cross University, Australia, who started the Mathematical Recovery Program (Wright, R. J., Martland, J., & Stafford, A. K., 2000; Wright, R. J., Stewart, R., Stafford, A., & Cain, R., 1998), provided me with the following data on some of the school systems he works with in the United States. Wright's Mathematical Recovery Program focuses specifically on children in the stage of the perceptual counting scheme. One school system in Arkansas reported that, over a four-year period, an average of 45% of their first-graders enter school in the stage of the perceptual counting scheme. In contrast, a school system in Wyoming reported that only 10% of their entering first graders are in the perceptual stage. However, another Wyoming school system reported that 61% of their children were in the perceptual stage and another 12% were yet to learn to count. Similarly, a school in Maryland reported that 60% of their entering first graders were in the perceptual stage and another 8% were yet to learn to count. Finally, in an assessment of the 1st Grade children in an elementary school that included children of faculty and graduate students of a Southern University as well as children from the general population of the city in which the University is located, approximately 30% of the children were in the perceptual stage (33). In the Southeastern region of the United States, especially in the intercity and rural areas, the percentages would be more compatible with the Arkansas data. So, my best estimate is that about 40% of the children in the United States enter the 1st Grade in the perceptual stage of the counting scheme or prior to that stage. Wright, himself an Australian, has extensive experience working in Mathematics Recovery in the United States. Of my estimate of 40%, he commented, "I think that is a good estimate for the number in the perceptual stage or lower, that is the children who can't yet count perceptual items. I think the percentage would be lower in Australia and New Zealand, say about 30%" (Personal Communication). The need for Wright's program is indicated by its widespread use. In a personal communication, Wright estimated that about 3000 schools are using his program in the United States, 1500 in England, and 4000 in New Zealand and Australia. Further, there are 272 schools using the program in Ireland with plans for at least that many more joining.

Transition to the initial number sequence. In a two-year teaching experiment, I taught three children who began the 1st Grade in the perceptual stage approximately 60 times over the two years. These children also participated in their regular mathematics classrooms. The children remained in the perceptual stage during their 1st Grade and transitioned into the figurative stage in their 2nd Grade. So, I expect that most of these children will enter the figurative stage sometime during the 2nd Grade (Steffe & Cobb, 1988). It wasn't until their 3rd Grade that at least one of them constructed counting on, the behavioral indicator of the INS. So, for the 40% of the school population who enter the 1st Grade having constructed only the perceptual counting scheme, it is reasonable to expect that the majority of them will construct the INS during their 3rd Grade [Wright estimated that from 5 to 8% might not be counting on by the 3rd Grade].

Based on other teaching experiments, my expectation is that a majority of these children will remain in the stage of the INS throughout the 5th Grade. The relative percentages are not certain, but because of the length of time and the great difficulties we had in both teaching experiments in engendering stage shifts, my best estimate is that approximately 30% of the children entering the 6th Grade will be in the INS stage in the construction of a genuine discrete quantitative measuring scheme. Wright's estimate was, "that about 30% of kids entering the 6th Grade in the US will only be able to count on" (Personal Communication).

The ENS Across Grades. Piaget, Inhelder, & Szeminska (1960) found that *only one child in ten* of those from six to seven years of age [1st Graders] have constructed operational measurement. So, I would expect that no more than one child in ten entering the 1st Grade will have constructed the ENS—10%. Piaget et al. (1960) used a 75% criterion concerning when operational measurement is constructed. With respect to this criterion, the authors commented that, "measurement in its operational form … is only achieved at about 8 or 81/2" (p. 126). As I point out below, the operational form of measurement entails iterable length units so, for *Swiss* children, I would expect that about 75% of them would have also constructed the ENS by what is the 3rd Grade in school in the US. But this estimate is overly optimistic for children in the US, if for no other reason than most of the approximately 40% of 3rd Grade children will have just constructed the INS. The others of the approximately 40% will remain in the figurative stage. Piaget et al. (1960) also found that *one-half of those* from seven to seven years six months [2nd Graders], attained operational length measurement (p. 114). Because the population in Switzerland is much more homogeneous than the population in the United States, it is implausible that 50% of USA 2nd Graders have constructed the ENS as I would expect in the Swiss population. My best estimate is that only 30% of the 2nd Grade population in the US will have constructed the ENS and that 50% of the US 3rd Graders have constructed at least the ENS. Based on a past teaching experiment (Steffe & Olive, 2010), I would also expect that by the 3rd Grade, the 10 percent of entering 1st Grade children who have constructed the ENS will have made progress and will be able to take three levels of units as a given in operating (Olive & Steffe, 2010). This 10% is included in the 50%.

Transition to the ENS. Approximately fifty percent of entering first grade children will have constructed the figurative counting scheme or the initial number sequence, where there is approximately an even split between the two. So, of children entering the 1st Grade in the US, my current approximations of how they are distributed among the number sequences follow.

PCS	FCS	INS	ENS
40%	25%	25%	10%

By the second semester of the 1st Grade, I expect that the children who start the 1st Grade in the figurative stage will have constructed the INS and that most of the children who start the 1st Grade in the stage of the INS will remain in that stage or will progress to a transitional stage in the constructive process that I call the tacitly nested number sequence, a stage that I have not discussed in this paper (Steffe, 1994).

As noted above, I expect that 30 percent of the 2nd Grade population will have constructed at least the ENS, which includes those children who make progress beyond the ENS by the 3rd Grade. What this means is that 2/3rd of the 30 percent must come from the 25 percent of the population who enter the 1st Grade in the stage of the INS. Because there is approximately 40% of the children in the 2nd Grade who have constructed the FCS, there must be 30% of the 2nd Grade children who are in the stage of the INS. This 30% of the population is carried over from 1st Grade children who were in the INS stage. So, my current approximations of how children are distributed among the number sequences in the 2nd Grade follows.

FCS	INS	ENS
40%	30%	30%

By the 3rd Grade, I assume that the population is essentially evenly distributed among the INS and the ENS given that approximately 10% of the ENS children can take three levels of units as a given.

Length Measuring Schemes

To further justify the foregoing developmental analysis of discrete quantitative measuring schemes, in what follows I coordinate these schemes with Piaget, Inhelder, & Szeminska's (1960) analysis of stages in the development of length measuring schemes. Length is one of the properties of a continuous item of experience that is introduced by a knowing subject. The construction of length involves motion of some kind in that it entails an uninterrupted moment of focused attention bounded by unfocused moments. It might be a sweeping of one's hand through space, walking along a path, scratching a path in the frost on a window with a fingernail, or moving ones eyes over the trunk of a rather tall tree. Insofar that each of these motions have a beginning and an end they can be isolated from the rest of one's experiential field and, along with the sensory material from the visual, tactual, or kinesthetic mode, form what I call experiential continuous units. Although the path of the motion is usually thought of as the length of the experiential object, the visual perceptual records may be overemphasized. In that case where there is an awareness of the duration from the beginning to the end of the motion, then an awareness of the visual records along with an awareness of the records of the motion constitute an awareness of experiential length. If the experiential continuous unit can be re-presented in the absence of perceptual input, there would be an awareness of figurative length if the motion along with its trace could be re-enacted in visualized imagination. If the object concept becomes interiorized in the sense that the arithmetical unit is interiorized, then what I have called an awareness of figurative length would be reconstituted as an awareness of the length of the interiorized concept. For this reason, *I consider an awareness of length as a property of an interiorized object concept* that an observer would judge as a linear object.

If a linear object concept is at the same level of interiorization as a numerical composite and, further, if a child uses the linear object concept in generating an image of an experiential linear object, the child uses the records of experience that were recorded in the attentional pattern that was used in establishing an experiential linear object. There are three kinds of records that I point to. The first are records generated using sensory material in the visual channel. I consider using these records to generate an image in visualized imagination which produces an image of spatial extension that might be referred to as a *segment* that can be used in assimilation [recognition] in the way that a numerical composite is used in assimilation to produce experiential unit items that are instantiations of arithmetical unit items. The second kind of records are the records of motion that are produced by scanning experiential linear objects from the beginning to the end. When instantiated in visualized imagination, there would be an image of sweeping the segment. The third kind are records of the duration of the motion. So, I regard an awareness of the results of the motion through a temporal interval as an awareness of the length of a segment. An awareness of a segment, together with an awareness of its length, constitutes an *awareness of magnitude*. If the segment is unitized, I refer to an awareness of the length of the segment as an awareness of magnitude. In short, the magnitude of a stick is an interiorization of its length.

My hypothesis is that such a concept of length can be constructed without a unit of length measurement or a measuring process being available (Steffe, 1991). Martin's way of operating that was illustrated in the opening of the paper indicates a co-presence of length units and arithmetical units. In the later presentation of the figurative lot structure, I did not account for the intervals between the figurative unit items in those cases where the elements are arranged in an identifiable path. These intervals are not empty because they contain perceptual material on which to focus attention. Likewise, in a figurative path of segments,[24] where a segment plays the role of a figurative unit item, what is in-between the segments is not vacuous even if it is nothing but a mark as in Martin's case of the marked ruler. As I noted above, the property of a segment that I call length is an awareness of the scanning action over an image of the segment along with an awareness of the duration of the scanning action. If the child is aware of a figurative plurality of segments, then that awareness, when coupled with an awareness of the scanning action over successive segments, is what I mean by an awareness of figurative length of a row of segments. The construction of a figurative row of segments is meant to illustrate how children's construction of continuous quantity at the most elementary level involves operations that are also involved in their construction of discrete quantity.

The construction of continuous but segmented object concepts illustrated by Martin provides an opening to consider interiorized composite wholes comprising discrete items as partitioning templates. I can see no reason, however, why Martin's construction of a continuous but segmented continuous object involved the operation of subdivision using partitioning templates. Further, the construction of such complex object concepts as a row of telephone poles or a sidewalk[25] *does not account for the construction of a unit of length that can be used to measure unmarked linear objects*. Piaget, Inhelder, & Szeminska (1960) observed what they thought was a developmental lag in the construction of a unit of length.

> Unlike the unit of number, that of length is not the beginning stage but the final stage in the achievement of operational thinking. This is because the notion of a metric unit involves an arbitrary disintegration of a continuous whole. Hence, although the operations of measurement exactly parallel those involved in the child's construction of number, the elaboration of the former is far slower and unit iteration is, as it were, the coping stone to its construction (Piaget et al. 1960, p. 149).

So, a unit of length for Piaget et al. (1960) was an iterable unit. Based on the discussion of Martin's finding how long two pipe cleaners were, children conceive of length units long before they become iterable units. The realization that arithmetical units become iterable units at the state of the ENS might lead to the hypothesis that length units can become iterable units without "arbitrary disintegration of a continuous whole" or, in my terms, partitioning, intervening. This is indeed the case with respect to the construction of connected number sequences, which are constructed through generalizing assimilation of discrete number sequences (Steffe, 2010). But connected number sequences are not length measurement schemes. Length measurement schemes are anticipatory schemes in the sense that when such a scheme is used in the context, say, of finding the height of a rather tall white oak tree, there is an anticipation that the height of the tree can be mentally partitioned into length units using the measurement unit before measuring. Following Piaget et al. (1960), such anticipation already entails mentally projecting length units into the extent of the tree as a constitutive aspect of using the scheme in assimilation because, "the notion of a metric unit involves an arbitrary disintegration of a continuous whole" as cited in the above quote. That is, partitioning operations are constitutive operations of a length measuring scheme. In what follows, I use work from Piaget et al., 1960, to correlate

24 It is useful to think of a visualized image of an open polygonal path in interpreting what I mean.

25 A row of telephone poles is a generic "model" of a "continuous" structure where what is in between the telephone poles is a constitutive part of the structure. A sidewalk is another generic "model" of a "continuous" structure where what connects the segments of the sidewalk is a constitutive part of the structure.

steps in the construction of equi-partitioning operations with the construction of the number sequences (Steffe, 2010).

Preoperational Partitioning Behavior

Partitioning a continuous unit into a specific number of equal parts originates in the context of using numerical dyadic and triadic patterns to project units into a continuous unit. To illustrate my hypothesis about how an INS child would use a numerical composite for five in partitioning, a protocol extracted from Piaget, et al., 1960) is presented. A child SIM, who was five years and nine months of age and in stage IIB in Piaget's succession of stages, succeeded in sharing a circular cake equally among three children, but proceeded as in Protocol IV in the case of five children.

Protocol IV

> What he does is to cut a series of small slices and deals them out as he goes along, leaving an unused remainder. When given another cake and told to finish it all, he cuts it into seven successive slices and distributes the first five only. Later, he finishes with six parts, but he never succeeds in dividing into five (Piaget et al. 1960, p. 323).

In this example, SIM initially focused on making slices without coordinating the number of slices with the whole. When he was asked to focus on the whole, he made enough slices to exhaust the whole, but without coordinating the number of slices, five, with the whole. SIM definitely had a goal to share the cake among five children as indicated by distributing the first five of the seven slices. This goal would be made possible by the use of the numerical composite, five, in making slices of the cake. In actual partitioning activity, SIM surely lost his sense of the numerosity of the numerical composite, five. His goal was to make enough slices to distribute five. Initially, he may have intended to make five slices, but in actual partitioning, he focused on simply making slices to exhaust the whole, an activity that was set in motion by his goal to make five pieces.

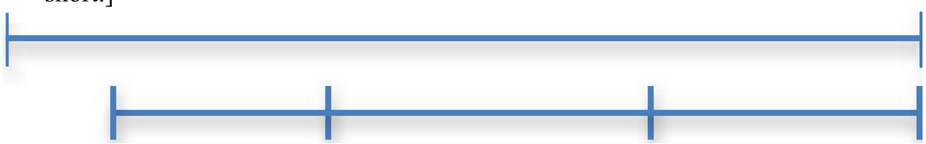
SIM's behavior is what I expect from a child who uses a *numerical composite* in partitioning, as opposed to a *composite unit*. In a numerical composite there is no composite unit containing its elements, so the composite whole is not an object of reflection for the child and the child is not aware of the composite whole as one thing. Rather, the child operates at the level of the unit elements. In a numerical composite, the child can produce the five cake pieces in visualized imagination, but cannot yet "hold them at a distance" and operate on these cake pieces by coordinating their size and shape with the whole they are to be part of. Similarly, when the child focuses on the whole of the continuous unitary item, the cake, he loses sight of the number of pieces, which is exactly what happened when SIM was asked to finish all of the cake (as indicated by cutting the cake into seven pieces and distributing five).

There was no indication that SIM was aware that the slices needed to be equal if they were to be fair shares, which is one of the seven criteria of operational subdivision articulated by Piaget et al. (1960). There was also contraindication that SIM was aware that five slices had to exhaust the whole, which is another of Piaget et al.'s (1960) seven criteria for operational subdivision. Further, although SIM did make slices of the cake, it is equivocal whether SIM's actions can be interpreted as meeting Piaget et al.'s (1960) first criterion of operational subdivision, which is that the whole is a, "divisible whole, one which is composed of separable elements" (p. 309). SIM made slices in action and, it would seem, without estimating where the cuts should be made prior to action. Had SIM made such estimates prior to action, it would have been a solid indication of projecting the elements of the numerical composite, five, into the cake prior to making slices. What he seemed to do was to use his numerical composite, five, in action when making the slices. So, it would seem that any length measurement activity that involves subdivision of a continuous whole in which an INS child engages would necessarily be preoperational with respect to length measuring schemes.

To corroborate my analysis that the partitioning behavior of INS children is preoperational, I turn to a study by Biddlecomb (1999) who analyzed the partitioning behavior of an INS child, Jerry, on a fractional task while Jerry was in his 4th Grade. Jerry was working with his partner, Adam, using a computer tool called TIMA: Sticks. Jerry was asked to make one-third of a unit stick. In Protocol V, Jerry drew a stick that he estimated as one-third of the unit stick and repeated it into a 3-stick [a stick consisting of three parts] that was shorter than the unit stick.

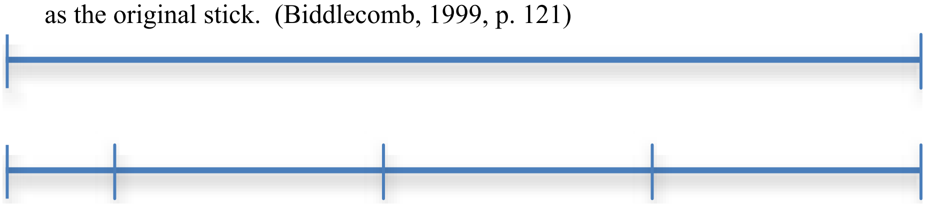
Protocol V

J: [Takes the mouse and draws his guess, repeating it into a 3-stick. His guess is too short.]



T: Is it exactly one-third?

J: Yeah. [Lines up the right end of the 3-stick with the original stick and then draws a small stick at the left end of the 3-stick so that the resulting total length is as long as the original stick. (Biddlecomb, 1999, p. 121)



Unlike SIM in Protocol IV who tried to make five slices of the cake in action, Jerry drew an estimate of one-third of the unit stick. To draw his estimate, Jerry would need to mentally mark off one of three parts of the unit stick, which indicates that he used his triadic pattern, three, in mentally marking the stick into three parts, which is compatible with SIM's sharing a cake among three children. After Jerry repeated his estimate into a 3-stick, he thought "it" was exactly one-third[26]. What he then did to justify why his 3-stick was exactly one-third was to align the right end-points of the two sticks and draw a small stick at the end of his 3-stick to complete a stick the same length as the unit stick. This behavior indicates that "one-third" did refer to the original unit stick as well as to his 3-stick. It also indicates that he was aware that the three parts should exhaust the unit stick.

In other protocols, if the three parts extended beyond the unit stick, he would cut off a small part of the three parts so the stick he made was the same length as the unit stick. He operated in the same way with "one-half" (Biddlecomb, 1999, pp. 116 ff). In fact, after Jerry had cut off one of the offending two parts to equalize a 2-stick he made and the unit stick, he thought that both parts of the altered 2-stick were one-half of the unit stick although they were not of equal length (Biddlecomb, pp. 119-120). It is very instructive to contrast the way Adam operated with Jerry's altered 2-stick to justify why the longer part was not one-half of the unit stick, with

26 At this point, "it" referred to the 3-stick [to "three"] as well as to the estimated part ["one-third"].

Jerry's belief that each part of the altered 2-stick was one-half of the unit stick. Adam pulled the longer of the two parts out of the altered 2-stick and placed it end-to-end with Jerry's longer part. The left end of Jerry's longer part coincided with the left end of the unit stick, so the two longer parts extended beyond the end of the unit stick. Adam said, "Nope", meaning that Jerry's longer part was not one-half of the unit stick (Biddlecomb, 1999, p. 119). Adam understood that if a part of a unit stick was one-half of the unit stick, then two of the parts must constitute the whole, which is an essential kind of an operation if a part of a stick is to be used as a unit to measure other sticks.[27]

Concerning Jerry's fractional activity, Biddlecomb (1999) said,

> Jerry's responses … might lead to the belief that he understood two of the criteria for a part of a whole to be one-third: that each part must be the same size and that all of the whole should be used. As we have seen earlier in his way of constructing half and third by cutting off the excess amount of his estimate, Jerry lacked coordination between the two criteria. He would lose the focus on the equality of the parts while trying to exhaust the whole. (p. 123)

Jerry's partitioning activity is compatible with the way SIM operated when sharing the cake among five people in that there was a lack of coordination between two essential aspects of operational partitioning. SIM did not coordinate the number of parts with the exhausting the whole of the cake in subdividing, whereas Jerry did not coordinate the equality of the parts while trying to exhaust the whole.

Using Composite Units Whose Elements are not Iterable in Partitioning

In an analysis of using discrete structures in subdividing continuous units (Steffe & Olive, 2010), I established that when a composite unit is used as a template for subdividing [partitioning] segments, five of the seven aspects of operational subdivision identified by Piaget et al. (1960, pp. 309–311) are satisfied. These are aspects 1–5, which I outline below.

1. First, children regard the continuous unit as a "divisible whole, one which is composed of separable elements" (p. 309) because the elements of the composite unit comprise the "separable elements" when *projected into the continuous unit*[28].
2. Second, children can partition the continuous unit into a determinate number of parts using a composite unit of specific numerosity.
3. Third, the whole of the continuous unit is exhausted and there is a coordination of the number and size of the parts with exhausting the whole of the continuous unit.
4. Fourth, children can establish a relationship between the number of parts and the number of cuts.
5. Fifth, children are sensitive to the equality of the parts.

There is an aspect of using a composite unit whose elements are not iterable in partitioning not mentioned by Piaget et al. (1960). Even though children might separate the parts of the continuous unit produced in partitioning, these children can reunite the parts into a continuous

27 Adam's number sequence was judged as explicitly nested.

28 The unitizing operation that constitutes the units of the composite unit is used in marking off segments of the continuous unit. I call this projecting the units of the composite unit into the continuous unit.

but segmented unit that is equivalent to the original continuous unit using the uniting operation. However, separating the continuous unit into parts and uniting parts together to form a segmented unit are sequential operations. As a consequence, the children may regard the continuous but segmented unit as a result that is unrelated to the continuous unit with which they began. In any event, these six aspects of partitioning are not sufficient for children to establish a length measurement scheme.

Only later, after the emergence of iterable units, does the possibility open for the children to regard the whole as invariant under partitioning and the sum of the parts to equal the original whole, which is Piaget's et al. (1960) seventh aspect of operational subdivision.

Equi-Partitioning Scheme

It is at the stage of the ENS that an *equi-partitioning scheme* emerges. In an equi-partitioning scheme, children can make an estimate of one of several equal sized parts and iterate the part to reconstitute an equivalent whole in a test of whether the part is a fair part. Protocol VI below occurred with two children, Jason and Patricia, during May of their 3rd Grade (Steffe & Olive, 2010, pp. 77-78).

Protocol VI: Equi-partitioning operations.

T: Let's say that the three of us are together and then there is Dr. Olive over there. Dr. Olive wants a piece of this candy (the stick), but we want to have fair shares. We want him to have a share just like our shares and we want all of our shares to be fair. I wonder if you could cut a piece of candy off from here (the stick) for Dr. Olive.

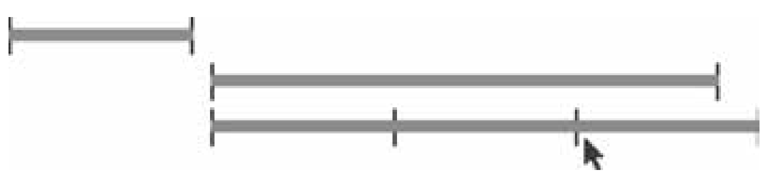
J: (Using Marks, makes three marks on the stick, visually estimating the place for the marks.)

P: How do you know they are even? There is a big piece right there.

J: I don't know. (Clears all marks and then makes a mark indicating one share. Before he can continue making marks, the teacher–researcher intervenes.)

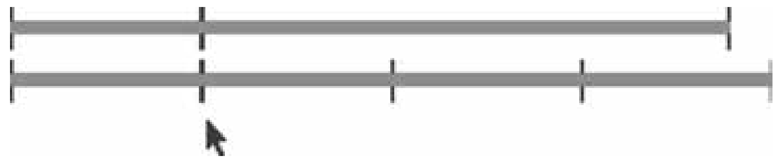
T: Can you break that somehow? (The teacher–researcher asks this question to open the possibility of iterating.)

J: (Using Break, breaks the stick at the mark. He then makes three copies of the piece; aligns the copies end-to-end under the remaining piece of the stick starting from the left endpoint of the remaining piece.)



T: Why don't you make another copy (This suggestion was made to explore if Jason regarded the piece as belonging to the three copies as well as to the original stick.)?

J: (Makes another copy and then aligns it with the remaining part of the original stick. He now has the four copies aligned directly beneath the original stick which itself is cut once. The four pieces joined together were slightly longer than the original stick.



Jason independently copied the part he broke off from the stick three times in a test to find if the three copies together would constitute a stick of length equal to the remaining part. This way of operating was crucial because it was the basis of my inference that he anticipated producing the three copies prior to their production. This anticipation would require him to *repeatedly use* the operations involved in making a stick in visualizing a stick, which is essential in iterating the stick. Comparing the three copies with the remaining part of the original stick does indicate that Jason took the three joined copies as a term of comparison; that is, as a unit containing three units that he could compare with the unmarked part of the original stick. This opened the possibility that he could unite a current copy of the part with those he had previously made. The possibility is confirmed when he made another copy and then aligned it with the remaining part of the original stick after the teacher suggested that he make another copy.

Jason's way of operating in Protocol VI was a modification of the operations that constituted his concept of four. In fact, the *result* of Jason's mathematical activity in Protocol VI can be regarded as a *connected number*, four. It is crucial to understand that Jason's independently contributed language and actions warranted imputing the connected number, four, to him. It is not a coincidence that the part that Jason broke off from the stick was a unit that he could iterate four times in a test to find if the part he made was a fair share of the original stick for two reasons. First, Jason's unit of one of his number sequence was an iterable unit. So, when he used his number concept, four, in partitioning the stick, the operations that he used to make the part originated in the operations that produce his unit of one. Second, six of the seven aspects of Piaget et al. (1960) operational subdivision were present in Jason's partitioning activity. The first five aspects listed above were certainly present as was the seventh aspect because Jason regarded the whole as invariant under partitioning and the sum of the parts to equal the original whole. The whole was invariant under partitioning because Jason knew that the part was too long. In that he iterated the part in a test of whether the part was a fair part, Jason certainly knew that the sum of the parts had to be equal to the whole.[29]

I consider the whole of Protocol VI as indicating an equi-partitioning scheme because I can infer a situation [Jason's situation], an activity [equi-partitioning operations], and a result [a connected number four]. This scheme is basic in the construction of length as a quantity in Thompson's way of regarding a quantity. But it is not sufficient because the units produced are yet to be constituted as iterable length units. The necessity to engage in equi-partitioning distinguishes discrete and continuous measuring schemes, but it does not separate them. This realization leads to a *reorganization hypothesis* that children's continuous quantitative measuring schemes can be realized as accommodations of their discrete quantitative measuring schemes just as we hypothesized (and corroborated) that children's fraction schemes can be realized as accommodations in their numerical counting schemes (Steffe & Olive, 2010, p. 1). In fact, I

29 Although not mentioned in the protocol, Jason kept making estimates until he was satisfied that the estimate was a fair part.

take a step further and claim that children's fraction schemes can be used as measuring schemes in a way that is analogous to the claim that children's number sequences can be used as discrete quantitative measurement schemes.

Fraction Schemes and Measuring Schemes

The partitive fraction scheme, the fraction scheme that follows on from equi-partitioning, is the first fraction scheme that I consider as a genuine fraction scheme (Steffe, 2010). The reason is that the equi-partitioning scheme, whose operations serve as assimilating operations of the partitive fraction scheme, satisfy six of the seven criteria of operational subdivision identified by Piaget et al. (1960, pp. 309–311)[30]. When it is a goal of a child who has constructed the partitive fraction scheme to mark off a unit part of a continuous whole, the child can disembed the part from the whole and iterate it to produce another partitioned continuous whole to compare with the original continuous whole in a test to find if the part is a fair part. So, in a task where it was the goal of Jason to mark off one-tenth of a segment, Jason marked of one of ten parts and iterated that part ten times to produce a segment of length equal to the original segment (Steffe, 2010, p. 100). He knew that the result was not only ten-tenths, but it was the measured length of the segment as well. Further, using a partitive fraction scheme, to find how much three of eight equal parts of a cake is of the cake, a "partitive fraction" child can partition the cake into eight equal parts, disembed one part and iterate it three times and call the result "three-eights" of the cake because it is three out of eight parts. The fractional meaning for "three eights" resides in the operation of projecting the three parts back into the eight parts, which relies on disembedding three parts out of the eight parts.

But if the "partitive fraction" child was given a part of the cake and told that the part was, say, five-eights of the cake, the child would be unable to engage in the operations to produce the whole cake. This indicates that "five-eights" does not refer to one-eighth five times[31]. That is, one-eighth is not an iterable fractional unit in the sense that five-eights stands in multiplicative relation to one-eighth so that five-eights is five times one-eighth. The partitive fraction scheme is yet an additive scheme rather than a multiplicative scheme. It might seem that I am being blatantly internally inconsistent when saying that a child who operates with the partitive fraction scheme can iterate one-eighth three times to produce three-eighths but that, given a part that is said to be five-eights of a fractional whole and asked to produce the fractional whole, the child is unable to partition five-eights into five parts to produce one-eighth because one-eighth is not an iterable unit. There are two reasons why I am pleading innocent. The first reason is that this is not something I just made up through a logical analysis while sitting at my desk in my office or at my kitchen table while staring out of the window at the Oak trees in my backyard. Rather, children taught me these distinctions while I worked with them in teaching experiments. It was through conceptual analysis of the video-recorded teaching episodes that I formulated the following analysis of the operations that I take as constituting the partitive fraction scheme. I start with the ENS because that is the scheme that is used in the construction of the partitive fraction scheme.

In the ENS, children can perform *progressive integration operations*. By an integration of two composite units I mean that the child, whose goal it is to find how many elements there are

30 The criterion that is not satisfied is the sixth criterion--the units produced by a partition can be taken as units to be subdivided further.

31 It indicates more in that "five-eights" would need to refer to a unit containing five one-eighth units that has been disembedded from a composite unit containing the disembedded unit as well as three more one-eighth units.

in the two composite units together, first unites the two composite units into a unit containing them, disunites each of the two composite units into their elements while maintaining an awareness of the two composite units as well as of their elements, and then counts the elements to establish their numerosity. The containing composite unit serves as background while the child is operating. Progressive integration operations are used when a child counts in the following way to find eleven and how many more are nineteen: "One more is twelve; two more is thirteen; three more is fourteen; …; eight more is nineteen. Eight more." In saying, "three more is fourteen." the child has integrated the next one after two more with a composite unit containing two more elements and construes one more than two more as three more. Continuing on in this way constitutes progressive integrations. It is also important to note that there may be another series of progressive integrations when the child produces "twelve" through "nineteen."

Progressive integration operations are built on one being an iterable unit and on using the uniting operation as an integration operation. So, when a child uses a composite unit whose elements are iterable units in partitioning a continuous unit and disembeds one of the units and appropriately calls it, say, "one-seventh", the disembedded part inherits its iterability from the iterable unit of one that was used to produce the one-seventh part. Further, if it is the child's goal to make, say, five-sevenths, the child can engage in progressive integration operations, "one-seventh; two-sevenths; …; five-sevenths". I refer to such an occasion as using one-seventh in *iterating* and have shortened it to say that the child iterates one-seventh without elaborating either on progressive integration operations nor on the source of the iterablilty of the part. Progressive integration operations are those operations that account for fractions as additive concepts.

When fractions are constructed as multiplicative concepts, any proper or improper fraction is constituted as a multiple of its unit fraction. Establishing fractions in this way involves taking three levels of units as a given in operating. To construct, say, eight-sevenths of a segment as eight times one-seventh of the segment, a child first partitions the segment into seven equal parts to establish seven-sevenths. To conceptually produce eight-sevenths of a segment, the child must posit a composite unit that contains the unit containing the seven-sevenths of the segment and another unit containing the remaining seventh and then integrate the two assemblages of units to produce a segment of length eight-sevenths. That is, the child must take a unit of units of units as a given in operating further. It is at this point in development that the child reconstructs partitive fractions as fractional numbers in that eight-seventh takes it's meaning from the fractional part of which it is a multiple. Eight-sevenths is eight times one-seventh and the latter iterated eight times is eight-sevenths. The fractional meaning of eight-sevenths is no longer directly dependent on its relation to the segment of which one-seventh is a part. The relation is inferential in that it can be established by reasoning of the sort, "This segment is eight-sevenths of the unit segment because it is eight times one-seventh of the segment." One-seventh is said to be freed from the unit segment of which it is a part and so it can be used as an iterable unit in the sense that the unit of one is an iterable unit. However, it takes three levels of units to establish a unit fraction as an iterable unit whereas two levels of units is sufficient to establish the unit of one as an iterable unit in the case of the ENS. My contention here is that *a measurement of what an observer would consider as an extensive quantity is a multiplicative object in the sense that a fractional number is a multiplicative object.*

Discussion

The central theses of the paper has been that human beings construct the quantitative properties of conceptual entities that are to be measured in the process of constructing the conceptual entities, the units used to measure the quantitative properties, and the measurement process used to measure the quantitative properties. The constructive process is an elongated process that occurs over years rather than over days or even months and is sensitive to the involved quantity. The necessity to construct operational subdivision introduces an aspect of length measurement schemes that produces a lag in their construction by children when compared to their construction of discrete quantitative measuring schemes. An essential aspect of continuous quantity that Thompson (1990) did not mention in his definition is that the magnitude being measured must be conformable to subdivision. Piaget et al. (1960) demonstrated that operational subdivision entails coordination of the number and size of the parts with exhausting the whole of the object undergoing subdivision, among other aspects. When children in the stage of the INS attempt to use their number sequence as a partitioning template, there is a lack of this coordination that is strikingly similar to the lack of the coordination in preoperational children reported by Piaget et al. (1960). This similarity corroborates my interpretation of the INS as signaling a stage in children's construction of genuine quantitative measuring schemes. But the INS is yet to become a genuine discrete quantitative measuring scheme because the units of the INS are not yet constructed as iterable. Still, at the stage of the INS, children have constructed a discrete quantitative property that I call figurative numerosity and do count to measure such numerosities. Similarly INS children have constructed length as a magnitude even though they are yet to construct a scheme to measure the magnitude because of the lack of operational subdivision. As Martin demonstrated, children can interpret segmented magnitudes as if the segments were discrete countable units and use their INS to count the units as if they were measuring without actually subdividing the magnitude to be measured. Still, a length unit to measure other length units is abstracted from the units produced by operational subdivision. In fact, a crucial aspect of transforming a counting scheme into a genuine length measuring scheme is that the child projects length units into the magnitude to be measured prior to measuring. It is particularly poignant that almost all of the children who are in the stage of the INS in the third grade will remain in that stage throughout their fifth grade and, correspondingly, be yet to construct operational subdivision. The lack of operational subdivision has alarming implications not only for the construction of length measuring schemes, but also for the construction of fraction schemes.

It is even more poignant when one realizes that 40% of the 1st and 2nd Grade population in the US are yet to construct the INS and that most of these children will construct at most the INS throughout their elementary school years. All of the data that I have presented in this paper is based on teaching experiments of duration two to three years, so genuine longitudinal attempts have been made to engender the construction of the INS by children in the two preceding stages and to engender the construction of the ENS by children in the stage of the INS. The presence of these children throughout the elementary school constitutes a crucial problem in mathematics education that remains unsolved. I consider the problem as exacerbated rather than solved by the current emphasis on outcomes-based education represented by the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). It is quite instructive to contrast what I have said about the lack of genuine measuring schemes in the case of INS children with a fourth-grade standard of measurement of the Common Core State Standards for Mathematics:

> Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; *lb*, oz.; *l*, *ml*; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two- column table. For example, know that 1 ft is 12 times as long as

> 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...

As a mathematics educator invested in quantitative reasoning, I can certainly resonate with a standard that emphasizes measurement units and relations among them. However, the standard is written as if it represents the classical dualism in Cartesian epistemology between mind and reality that has pervaded thinking in mathematics education throughout the last century and on into this century. I do consider it possible for INS children to construct magnitudes of time called "one hour", "one minute", and even "one second". I can also imagine such children counting from "one" to "sixty" while watching the second hand tick off seconds on a clock and call the result "one minute" similar to the way in which Martin counted to find how long two pipe cleaners would be if they were placed end-to-end. But that does not constitute operationally subdividing a magnitude called "one minute" into sixty parts and iterating one of the parts sixty times to produce "one minute". One could also easily imagine similar activities involving meters and centimeters, or liters and centiliters. But the reasoning that is involved in establishing relations between centimeters and kilometers or between hours and seconds involves operational subdivision that is made possible by taking three levels of units as a given in further reasoning using intermediate units. So, it is crucial to interpret what it might mean when we say that children are to; "Know relative sizes of measurement units within one system of units…" in terms of the measuring schemes and levels of units that are available to children.

Establishing that one foot is twelve times as long as one inch can be based on the equipartitioning scheme if an ENS child marks off one of twelve parts of an unmarked foot ruler, calls the part "one inch," and then iterates the part twelve times to produce a connected number, twelve, that can be regarded as a foot ruler segmented into inches. At this point the child, having constructed the ENS, could disembed any one of the twelve parts from the foot ruler and establish that one inch is one-twelfth of the composite unit comprising the twelve inches and that one-twelfth foot iterated twelve times is one foot as suggested in the standard. The child could also iterate 1/12 foot three times and judge the result as 3/12 foot in relation to 12/12 foot. The child could also establish 9/12 foot as the complement of 3/12 foot in 12/12 foot because each composite part could be disembedded from 12/12 foot and joined together to comprise the 12/12 foot. Given these operations, it would also seem as if the child could judge that 3/12 foot is also 1/4 foot. But that would entail establishing a composite unit containing four composite units each of which contains three units of 1/12 foot. In other words, it would entail the child anticipating that the unit containing the twelve unit fractions could be partitioned into four composite parts each of which contained three unit fractions, 1/12 foot. That is, it would entail the child using the structure of a unit of units of units in reasoning, which is a stage beyond the ENS—the stage of the generalized number sequence [GNS] (cf. Olive & Steffe, 2010). Further, to establish 1/12 foot as an iterable fractional unit so that, say, five feet could be established as 1/12 foot sixty times or that five feet seven inches could be established as 1/12 foot sixty seven times involves three levels of units. Finally, to measure in terms of feet and inches or in terms of yards, feet, and inches entails constructing *composite units* as iterable units. So, to establish a system of measurement units seemingly entails taking three levels of units as a given, which is a stage in reasoning with systems of units one stage above the ENS.

In the opening of this paper, I argued that the operations that generate an awareness of discrete quantity are not of a different genre than the operations that generate an awareness of length, distance, weight, area, volume, capacity, angles, temperature, time, and speed. Throughout the paper, I have extended that argument to the operations that generate an awareness of fractional quantity. Embedding the construction of discrete quantitative schemes and the construction of fraction schemes in the construction of measuring schemes *in appropriate ways* serves to integrate the study of quantity rather than separate it into discrete and continuous quantity. In the paper, I did not discuss what an awareness of fractional quantity entails nor did I discuss what an awareness of distance, weight, area, volume, capacity, angles, temperature, time, and speed entails. Further, I did not discuss the construction of schemes children use to measure an awareness of these quantitative properties. These are very complex matters that require continual intensive conceptual analysis of children's quantitative reasoning. Rather than be definitive, what I have tried to do is to provide an opening for an extensive research program into children's construction of quantitative schemes and the quantitative reasoning they make possible.

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