THE ADDITION AND SUBTRACTION OF SIGNED QUANTITIES

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Abstract

In the middle grades, students are expected to work with signed quantities, which take on both positive and negative values. I begin by outlining some important conceptions of quantity, addition, and subtraction that would help students operate successfully in their mathematics and science classes in middle and high school, particularly when these signed quantities are involved. I then outline the types of concepts of quantity, addition and subtraction that students have already developed in a natural number context. I use findings about four participants in a recent teaching experiment (Ulrich, 2012) in order to highlight key transformations in thinking that students make when constructing sophisticated signed quantity concepts. Namely, students will need to assimilate changes and comparisons in quantity as signed quantities and then mentally operate on these quantities to construct robust additive structures. In addition, students will become increasingly aware of the role of the reference point, often 0, in these structures as well as the role of additive inverses. At the end of the paper, I outline further research needed into, among other things, the development of the usual addition and subtraction notation.

In the US, curriculum recommendations often have students formally encountering signed quantities for the first time in fifth grade (Georgia Department of Education, 2008; National Council of Teachers of Mathematics, 2000, 2006) or sixth grade (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010), with addition and subtraction of signed quantities beginning in seventh grade (GA DoE, 2008; Natl. Governors Assn., 2010; NCTM, 2000, 2006). In all four of the standards documents I have cited so far, the 2000 Standards, NCTM Focal Points, Georgia Performance Standards, and the Common Core Standards, seventh grade is the last place where negative numbers are explicitly mentioned. However, we know that negative numbers do show up in mathematics and science courses in both middle and high school. In addition, the way that addition and subtraction signs are used changes in the secondary years, most noticeably when working with variables that represent signed quantities. For the majority of this paper, I will discuss how the earlier work with signed quantities in fifth through seventh grade can support the more sophisticated way in which students will need to work with quantity, addition and subtraction. At the end of the paper, I will also give some examples from a recent teaching experiment that highlight changes in the way that students assimilate signed quantities.

Where Do Students Need to Go?

I will now explore the question of what kinds of quantities we would like students to be able to work with and what meanings for addition and subtraction we would like students to have by the end of middle school, and I will explain how we can support students in reaching these goals during integer instruction in the early secondary years.

New Kinds of Quantities

The Common Core Standards (Natl. Governors Assn., 2010) give four examples of signed quantities that students could work with: “temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge” (p. 43). In the first two examples, virtually any addition situation would involve working with changes in values of the quantity. For
example, if a student were to attempt to develop a problem situation in a context of temperature involving the sum of two signed numbers, such as (-38) + (+12), then the second addend is almost certainly going to be a change in temperature, not a reading on the thermometer. An example of a question would be the following: *The temperature outside is 38 degrees Fahrenheit below 0 in the morning, but increases 12 degrees Fahrenheit by 4pm. What is the temperature at 4pm?* The second addend does not represent a reading of a thermometer in the same way the first addend and sum could. We can come up with addition situations that do not involve an explicit change, such as the combination of charged particles: 4.6 moles of positively charged sodium ions and 0.5 moles of negatively charged chlorine ions are combined in a neutral solution. What will be the overall electrical charge of the solution (in faradays)? However, many situations involving addition of signed quantities do require operating with changes in quantity.

In fact, in all of the first three examples, every signed quantity can be thought of as a change in quantity. This is perhaps most easily seen when working with another common signed quantity, directed distance from 0 when carrying out trips on a number line. If you are modeling (+3) + (-7) on a number line, then your starting position could be +3, or you could think of your starting position as the reference point and then see the +3 as representing an increase of 3 units (see Figure 1).

![Figure 1](image)

*Figure 1.* Two representations of (+3) + (-7) = (-4).

Having students work in contexts where all values represent changes in quantity has many benefits. One is that the commutative property of signed addition becomes much more intuitive when the student is able to interpret signed numbers as changes in quantity. Students tend to intuit, purportedly based on their experiences, that the order of increases and decreases is irrelevant when a series of increases and decreases is made. However, the situation is less intuitive when some of the signed values represent positions: Recognizing that starting at +3 and decreasing 7 gets you to the same value as starting at -7 and increasing 3 requires greater reflection on the situation. In addition, when working with changes, you can have an unspecified reference point. When students eventually work with vectors and matrices, the ability to deal with an unknown reference point is key. In addition, when interpreting slopes and rates of change in functions, the actual values of the underlying quantity are not as important as the way in which those values change.

This leads into an important conception of signed numbers that has helped orient me as a teacher and researcher: the idea of signed numbers as one-dimensional vectors. One of the key

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1 By unspecified reference point, I mean that you know what relationships the reference point has to the changes in quantity, but its value in terms of the underlying quantity is unknown. For example, if you put 30 cents in a piggy bank and take out 40 cents, you can figure out the overall change in the piggy bank’s value without knowing how much was in it originally.

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models used for representing signed quantities is movement on a number line. In this model, arrows represent signed quantities. This makes more explicit the vector nature of these signed quantities. Moving towards a vector conception of number is important for several reasons. First, students’ ideas of addition and subtraction should be easily transformable into corresponding vector notions for students continuing on in higher science or mathematics. Second, a number line model involving one-dimensional vectors will provide an opportunity for students to work with the number line, which is clearly important in mathematics and science yet tends to trip students up. This is in part because the number line forces the student to explicitly differentiate between the value of a number, which represents the directed difference from 0, and the transformation that brought you to that number, which represents the directed difference from the previous value. For example, in Figure 1, a student may want to label the point at -4 as -4 because of its position in relation to 0 or as -7 because it is the endpoint of a -7 translation. Third, just as vectors are equated with transformations at the post-secondary level, we want students to gain experience in reifying signed numbers as transformations (i.e., constructing transformations as mathematical objects that can be mentally operated on). The use of arrows on the number line as a model for integer operations provide opportunities for this kind of abstraction. Fourth, it seems reasonable to assume that the cognitive operations necessary for assimilating changes in quantities as mathematical objects would be necessary for such common situations as operating with changes in function values to form conceptions of rate, slope and derivative. Hence the focus on developing integers as one-dimensional vectors has both a theoretical and pedagogical foundation.

Note that in the same situations in which changes in quantity are themselves a quantity that is operated upon, such as defining average or instantaneous rate of change, these changes in quantity are usually determined through subtraction, through taking the differences of two values of the underlying quantity. Therefore, in all of these situations, a students’ ability to reflect on differences and operate with difference is important. Students can gain experience reasoning about differences in natural number contexts (Thompson, 1993), but the development of signed quantities offers another good opportunity, since a signed quantity can be built up out of the directed differences of two unsigned quantities or two signed quantities. In Linchevski and William’s work (1999), they try to engender construction of integer quantities out of the directed differences of dice rolls. Similarly, in my research, I used a card game in which students construct integer quantities out of pairs of natural number values. Initially, each student drew a number card (the base quantities) and the holder of the winning card got the difference of the cards’ values added onto their score. So far we are still dealing with natural numbers. However, once the students got the hang of the game, we just kept track of how much each player was winning or losing by instead of their actual score. Now the students were constructing integer quantities by keeping track of the directed difference of the numbers on the cards As well as the sums of these directed differences.

![Figure 2](image.png)

**Figure 2.** The card game.
So far I have discussed the types of quantities students will need to construct for upper-level mathematics, namely, additive transformations of (oftentimes continuous) quantities and additively derived quantities (often differences). Now I will switch to discussing how we might reconceptualize addition and subtraction when working with these new quantities.

**New Addition and Subtraction Concepts**

In dealing with changes in quantity, addition becomes a composition of transformations (Thompson & Dreyfus, 1988; Vergnaud, 1982). In particular, if I am focusing on situations in which signed quantities are referring to changes in quantity, students will need to construct an addition scheme that assimilates compositions of transformations as addition. With signed quantities in general, and with quantities that represent change in particular, subtraction is defined with respect to addition. In fact, there are two ways in which subtraction is generally defined. The first is subtraction as the determination of a missing addend, which I will call MAS. MAS is equivalent to thinking of subtraction as finding the directed distance between two values or how to get from one value to another. The second definition equates subtraction with addition of the additive inverse of the subtrahend, which I will call AAIS.

Both of these characterizations contrast with the other definition of subtraction, “take-away” subtraction, which we often use in both whole number and integer instruction. Subtraction as “taking away” is useful in many situations, but is not a useful conception when working with signed quantities, and certainly not when working with vectors. Consider, for example, the frequently encountered use of subtraction to denote the distance between positions (here the absolute value of a difference is the usual notation). There is no sense of anything being removed in this situation. It is true that “take-away” situations with natural numbers can be fairly easily reformulated as situations of a decrease, so that students may be able to assimilate situations with a downward movement, a decrease, etc., as subtraction situations, but this reformulation of “take-away” subtraction still would not be useful in making sense of subtraction of a negative number.

Briefly, I will give an example of MAS and AAIS in two dimensions. This is to elucidate the relationship between them for the reader. I did not use two-dimensional situations with my participants. Note that given the graphical representation of two two-dimensional vectors, \( \mathbf{a} \) and \( \mathbf{b} \), \( \mathbf{a} - \mathbf{b} \) generally represents one of two different vectors. The first possibility is that \( \mathbf{a} - \mathbf{b} \) represents the vector that is composed with \( \mathbf{b} \) to result in \( \mathbf{a} \), represented by putting \( \mathbf{a} \) and \( \mathbf{b} \) “tail-to-tail” and then drawing an arrow from the endpoint of \( \mathbf{b} \) to the endpoint of \( \mathbf{a} \), as in Figure 3a. The other possibility is that \( \mathbf{a} - \mathbf{b} \) represents the composition of \( \mathbf{a} \) and the additive inverse of \( \mathbf{b} \), represented by the arrow formed by drawing \( \mathbf{a} \) and \( -\mathbf{b} \) (\( \mathbf{b} \) pointing in the opposite direction) “head-to-tail” and then connecting the tail of \( \mathbf{a} \) to the head of \( -\mathbf{b} \), as in Figure 3b.

These two vectors are isomorphic, but they come out of different characterizations of subtraction: MAS and AAIS, respectively. Using Figure 3, the recognition of a logical necessity that the two vectors always be isomorphic could follow from the application of various geometric theorems. For example, as in Figure 4, I could start by drawing \( \mathbf{a} \), and then draw both \( \mathbf{b} \) and \( -\mathbf{b} \) in the same relationship to \( \mathbf{a} \) as in Figure 3. Next, I could note that, just as \( \mathbf{b} \) translated by \( \mathbf{a} \) (shown as a dotted arrow in Figure 4) is parallel to the original \( \mathbf{b} \), \( -\mathbf{b} \) will be parallel to \( \mathbf{b} \). Therefore, we could apply (1) the congruence of alternate interior angles (angles 1 and 2) formed by a transversal that cuts parallel lines, and (2) the fact that if two sides and the angle between those sides in one triangle are congruent to two sides and the angles between those sides in a second triangle, then those triangles are congruent.
Figure 3a. MAS example

Figure 3b. AAIS example

Figure 4. Congruence of $a + (-b)$ and $a - b$. 
Regardless of the method used to establish it, the logical necessity of isomorphism is not immediately obvious for the learner. In a one-dimensional situation, like the number line, the corresponding recognition that MAS and AAIS interpretations result in the same answer to a problem is equally, if not more, profound. In the one-dimensional situation (assuming the learner is unfamiliar with the two-dimensional situation), the recognition that the magnitudes are the same would seem to follow from a decomposition of the subtrahend, minuend or difference, depending on which of these has the largest magnitude, but then an additional analysis would be necessary to recognize that the directions of the differences would always match up. In fact, my own sense of logical necessity derived from breaking up all possible subtraction problems into several cases (the signs of the minuend and the subtrahend are the same or different, if the same, I have different cases depending on which magnitude is larger). Hence, the logical necessity of MAS and AAIS resulting in the same difference is less intuitively obvious for me in the one-dimensional than the two-dimensional case.

In many models of integer subtraction, a subtraction sign is taken to be equivalent to an addition sign followed by the additive inverse of the subtrahend (AAIS). This is probably because this view of subtraction seems to more easily match up with AAIS in that a take-away situation can often be reconceptualized as the composition of a positive change in quantity and a negative change in quantity, i.e., it can be reconceptualized as an addition of an additive inverse. For example, “If Verna has 9 apples and eats 3, how many are left?” is certainly a “take away” situation. It could also be thought of as a (prior) gain of 9 apples followed by a loss of three apples with a resulting net gain of 6 apples. In fact, Bob Moses’s Algebra Project was the only place I found subtraction foundationally characterized as MAS (Moses & Cobb, 2001). In the Algebra Project curriculum, the context was subway trips and integers represented the directed difference from one stop to another.

Regardless of which meaning of subtraction is taken as basic, we need students to be able to use them interchangeably. Therefore, students should have experiences with both characterizations of subtraction, and, ultimately, we would want students to develop a sense of why MAS and AAIS are equivalent. This equivalence is tied into a larger realization that subtraction situations and addition situations are both additive in nature. That is, the underlying additive relationship in either type of problem can be conceptualized as addition or subtraction, just as a natural number subtraction problem can be easily mapped to a missing addend situation.

In AAIS, the concept of an additive inverse is crucial. In fact, several authors (Thompson & Dreyfus, 1988; Flores, 2008) discuss the importance of constructing the additive inverse as a mathematical object. In my dissertation study, I also found that the role of the additive inverse is important for student-generated solutions to addition and subtraction tasks involving signed quantities. I will discuss this finding in more detail later.

**Where Are Students Coming From?**

In this section, I use the language from researchers and standards documents to illustrate some questionable assumptions many mathematics educators have about learning to add and subtract signed quantities. Although I use excerpts from the Common Core Standards, I want to stress that the Standards document gives other expectations for sixth and seventh grade as well, and I commend the writers for the thought and detail with which they outlined important aspects of signed number conceptions.
The Nature of Quantities
The first assumption I sense in the Standards relates to the nature of signed quantities. The Common Core State Standards (Natl. Governors Assn., 2010) call for sixth-grade students to “extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates” (p. 43) and “understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge)” (p. 43). In all of these quotes, a signed number represents a point on a number line or a value of a familiar quantity. Certainly these understandings of signed number are important. However, contexts involving addition or subtraction of signed numbers often also involve operating on changes in quantity and/or operating on differences of quantities, as discussed earlier.

Vergnaud (1982) carried out a study with a large group of elementary-aged children and found that natural number addition and subtraction problems in which both addends were changes in quantity were the hardest for students to solve. Thompson (1993) showed that interpreting a comparison of differences is also difficult for some fifth-grade students. Therefore, we cannot assume that all signed quantities are created equal. Students will have the most trouble with changes and differences in quantities and so need additional experiences with both.

Note that in the second quote from the Common Core Standards, positive and negative numbers describe the same quantity. In fact, some students will initially construct positive and negative quantities that are related by context, but are not seen as part of one overarching signed quantity. To get a sense of how this is possible, consider the chip model (e.g., Flores, 2008) for teaching addition and subtraction of signed numbers: The positive numbers and negative numbers are presented as representing two separate quantities, the number of black chips and the number of red chips, respectively. These quantities are linked by the ability to add or take-away zero pairs of equal numbers of both quantities, but it would be hard to verbalize a meaning for the overall quantity that the positive and negative numbers are describing. In contrast, in the context of elevation above/below sea level, there is a clearer unitary signed quantity, elevation, that is evaluated with reference to sea level. However, for students who are initially operating in the sea level example, there may be a tendency to focus on the elevations below sea level and above sea level separately, as with the chip model. Peled (1991) discusses students tending to use separate rules of actions depending on which of these two quantities, positives or negatives, that they begin their situation in: “There are two worlds: a positive world on the right and a negative world on the left” (p. 147). In addition, Janvier (1985) briefly notes that at one time in the history of mathematics, the real number line was split into two, with the positive portion separate from the negative portion even by experts. Therefore, the construction of a signed quantity is not trivial for most students, but involves a progressive awareness of the relationships between the related, but separate, negative and positive quantities, including their shared reference value. I will elaborate more on this process when I discuss some of my most recent findings.

Addition and Subtraction with Natural Numbers
The second assumption I sense in various Standards documents and some research literature is about the nature of students’ addition and subtraction schemes on entering seventh grade. In the Common Core Standards (Natl. Governors Assn., 2010, p. 48), students are expected to “Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers” (p. 48). Other standards documents, including the NCTM (2006) Focal Points use similar language: “Students extend understandings of addition, subtraction, multiplication, and division, together with their properties, to all rational numbers, including negative integers” (p.
19). I also found the same language in various researchers’ work (Linchevski & Williams, 1999; Peled, 1991). The use of the word extend implies that the same ideas of addition and subtraction that have worked for the students in the past will work with signed numbers. In fact, there are ways that students think about addition and subtraction that are not easily generalizable to a signed number context. In the elementary grades, addition is often characterized as increasing the amount in a set, and subtraction is often characterized as taking away items from a set or decreasing the amount in a set. Hence, for some students, addition and subtraction are actions as opposed to indications of a sum or difference structure. This can even be true for students who have a fairly sophisticated understanding of how the quantities in sums and differences are related. For example, all of my participants had constructed an explicitly nested number sequence (Steffe & Olive, 2010) and were aware that $24 - 7 = 17$ implies that 7 and 17 are both subsets of 24 and, taken together, give a set of size 24. Hence they understood the necessity of why $24 - 7 = 17$ implies $7 + 17 = 24$ and $24 = 17 + 7$. However, even for my participants, the notation of addition and subtraction does not necessarily indicate the presence of these sum and difference relationships. Addition and subtraction were still actions to be carried out. Thompson (2011) discusses the need for students to have more experience operating on and reasoning about notated sums and differences ($a + b$ or $a - b$) without carrying out the operations needed to determine a numerical value. Given the opportunities to reason about notated sums and differences, students might indeed come to middle school with more structural conceptions of addition and subtraction, particularly with respect to the notation. Also, I should point out that although many elementary school students use subtraction to solve for a missing addend or to compare the size of numbers, I have found the “take-away” characterization of subtraction to be the default and dominant characterization of subtraction for participants in my pilot study and dissertation study (Ulrich, 2010, 2012). In fact, I was unsuccessful with 5 of my 6 participants in engendering the assimilation of combination situations as sums or missing addend situations as differences, because of the tenacious link between addition and increasing and subtraction and decreasing in their ways of thinking.

What Happens Along the Way?

I will now use data from a semester-long constructivist teaching experiment (Steffe & Thompson, 2000) to highlight two aspects of the students’ construction of additive relationships between signed quantities.

Quantification of Changes/Differences

One student, Lily, came into the signed quantity section of the teaching experiment with very sophisticated whole number concepts. In addition, she seemed to be able to assimilate directed differences as quantities without any difficulty and dealt well with complex additive situations. She did struggle to communicate about and make explicit her thinking at times, but she always seemed to grasp the quantitative relationships at play. For a few sessions, she was paired with Justin, who also had strong whole number schemes. However, his additive schemes seemed to be more operational than structural. That is, he had an intuitive sense of quantities and quantitative relationships, but he did not assimilate situations with a sum or difference structure at the beginning of the experiment. By the end, he made immense progress, probably indicative of the fact that he had the operations available to construct signed quantities, directed sums, and directed differences, but did not have practice reflecting on sums and differences.

In the following protocol, the students were given a list of weekly weights and asked to

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2 The names of all participants are pseudonyms.
find the week when the biggest change in weight happened. This excerpt is from the ensuing discussion about how they each got their answer. Throughout the protocols, J represents Justin, T represent the teacher/researcher, and L represents Lily.

Protocol 1. Justin and Lily talk about differences.

J: I added, I subtracted, I added [to get from] that [Initial Weight] to that [Week 1]. That was only 7. I added 5 to that to make it 40. I added 8. I knew that would be 13. Then 148 minus 139 is only 9. And then 239 plus 1 is 40. Then minus 2 is 140. So that one [Week 3] was when he increased the most.

T: Is that the same way you were thinking about it, Lily?

L: Kind of. I was looking at it and some of them weren’t very far apart by pounds, like maybe 2 or 3 pounds. But this one has the biggest change. So that’s kind of how I got my answer.

I have bolded text in the protocol in order to highlight how the two students discussed differences. Justin uses action language throughout. He identifies the differences as describing his actions. Lily, on the other hand, refers to the differences as nouns, as quantities. In fact, she had a much easier time answering my questions about these weight changes than Justin. Later in this same teaching episode, I gave them both a sequence of numbers and asked them to continue the pattern. The sequence was 3, 4, 6, 10, 18, ..., which is exponentially increasing. Both students eventually solved the task, but Lily immediately notated the differences on her paper and referred to them in her explanation. Justin, on the other hand, was able to construct all of the necessary relationships, but had a much harder time keeping track of quantities, getting the answer, and explaining the answer. I think that Lily’s spontaneous use of notation signals her greater awareness of the differences as quantities that are themselves related by quantitative relationships, in this case, a multiplicative relationship.

Justin does struggle for the next few teaching episodes in constructing a signed sum. In the card game, time and again, he interpreted values in the third column of his score sheet (see Figure 5) as representing just total wins or just total losses. Hence in Round 3, he thought of won 15 as representing his previous wins, the 9 as representing his previous losses, and the won 85 as representing the new win that needs to be taken into account. Given that interpretation, he did correctly calculate won 91 as his total score. However, he would go on to then treat that score as just a record of his wins later on. But the very fact that he treats the other two column 3 scores as just an accumulation of wins or an accumulation of losses while realizing that he needs to add both together to get his new column 3 score shows his confusion about the quantities at play. This was not simply an issue of understanding the situation because he would sometimes give explanations about the quantities that made clear the relationships. In addition, he started writing winning by or losing by in front of the running total scores to differentiate them as quantities from the won and lost round scores, but would still treat the Column 3 scores as representing just wins or just losses from time to time even after that switch. I would posit that, while Justin can calculate the result of combining two changes in quantity, he is very much attending to the actions implicit in the changes themselves so that when he gets the sum, he is too caught up in action to reflect on his operations and develop a sense of the quantitative nature of the sum.
Figure 5. Re-creation of Justin’s score sheet for the card game.

When we switched to a context in which the value of the piggy bank is being increased or decreased, Justin initially misinterprets the sum of changes in this context as well. After determining the sum, he then reinterprets it as one of the original changes and combines it unnecessarily with the other change. In Protocol 2, M represents Michelle, another participant.

Protocol 2. Justin conflates with money.

[The problem: Justin adds 48 cents to the plate that represents a piggy bank. Michelle removes 20 cents from the plate. After both of their actions, will the total value of the plate increase or decrease and by how much?]

J: There will be more in the plate because it’s 48, you subtract 20 from that, it’d be 28. It’d be 8 more than you had.

T: How much more is going to be on the plate?

M: 28?

T: You’re saying 28, it wouldn’t—

J: I think it’s 8 because she took away 20 and if you’re adding 20, it’d be the same amount as you had and the extra 8 would be 8 more cents than you had.

T: How much did he add there then, though?


Justin does this again a couple more times, but later in this teaching episode, he seems to have an ah-hah moment in which he replays through the actions in his head and corrects himself. In fact, we can see in Protocol 2 that despite the fact that he is using the wrong values, his reasoning to find the second sum involves both a reference to an additive inverse, “she took away 20 and if you’re adding 20…,” and to 0, “it’d be the same amount as you had [to begin with].” As I expand upon in the next section, I hypothesize that this increased attention to 0 and additive
inverses is what allows the construction of a true signed quantity that is not divided into separate positive and negative worlds.

Another participant, Adam, frequently conflated the sign of the second addend with the sign of the sum in signed addition situations. I believe that he, like Justin, could not curtail the actions implied by the situation enough to let him step back from his mental (or physical) reenactments and reflect on the numerical relationships between the values he was working with. His actions and the outcomes of his actions become intuitively linked in a way that caused him to confuse quantities and interfered with any attempts to analyze the additive relationships between quantities. In Adam’s case, I do not think that reflecting on the additive relationships between the signed quantities was within his zone of potential development at the time of my study. He was the only participant out of the four who was not able to assimilate situations with three levels of units. Two levels of units is sufficient for him to assimilate the subset structure that underlies addition and subtraction of unsigned quantities. However, unlike with unsigned quantities for which the addends are always complementary subsets of the sum, a signed sum can correspond to one of the complementary subsets in the underlying unsigned additive relationship (see Figure 6). Given an unsigned additive relationship, we can transform it into a signed additive relationship given at least two additional pieces of information. For example, if we know which unsigned quantity corresponds to the sum and what the reference point is, which is equivalent to knowing which quantity corresponds to the sum and the directionality of the sum, then we can determine the signed additive relationship. Figure 6 illustrates all possible signed relationships that can correspond to an unsigned additive relationship. My hypothesis is that this added level of uncertainty about the quantitative relationship in a signed sum means that a student needs to be able to build a three-levels-of-units assimilating structure to reflect on the actions that result in signed sums.

![Diagram](image)

*Figure 6. Six signed additive relationships corresponding to an unsigned one*

**Adding and Subtracting Signed Quantities**

As I discussed earlier, students may initially treat positive and negative values as representing two related, but distinct, unsigned quantities. For example, in the card game, sometimes the participants would refer to their points and the other player’s points as the two quantities in the situation instead of referring to one signed quantity that represents their relative score (with respect to the other player). Usually the distinction between positive and negative values is less pronounced, but it does leave its mark.
A more subtle example of how a potential signed quantity exists as two related quantities lies in the students’ initial understanding of addition situations. Initially, when the students had less sophisticated signed quantities, they would explain the sign of the answer by comparing the sizes of the increase and decrease: “Because mine, it has more than hers. Because my number’s bigger, and it’s a decrease.” However, when the students gave explanations about the size of the answer, they would describe a process of elimination. For example, if I asked why a student subtracted to determine \((-2000) + (+350)\), the answer would be that “if you added, then it would be like you had another decrease.” As another example, in a missing addend situation, Michelle and Justin had agreed on the sign of the answer, tried subtracting to find the magnitude, and recognized that the answer would not make sense, so added instead. Justin explicitly said that he added because subtracting did not work. In these cases, in order for the students to judge whether an answer made sense, they had to have at least a nascent sense of how the quantities are related, but the students would nonetheless guess and check to determine which operation would give a reasonable magnitude.

The next step in reasoning can be seen emerging in Protocol 2. Here both 0 and an additive inverse can be explicitly referred to by the student. In Protocol 2, Justin talked about what he would need to do to have “the same amount as you had,” which is his indefinite reference point. Of course, he needed to use an additive inverse to get back to his reference point, so by attending to how to get back to the reference value, he will automatically begin focusing on how he can decompose, in this case, one of the addends to give him the part below the reference value and the part above the reference value. This ability to talk about the concepts underlying the reference quantity and additive inverses allows the student to reason within the situation to determine the size instead of just guessing whether to add or subtract. There is a necessity to the answer that was not there before. I also believe, based on my work with Justin and Michelle, that once a student begins to attend to 0 and additive inverses, the conversion from two related quantities to one signed quantity will quickly progress. By the end of the teaching experiment, three of the four participants, all but Adam, were able to reason through 0 and use additive inverses to determine answers.

The number line representation of trips up and down a ladder allowed the students to be even more explicit in their explanations of the subset relationships between addends and sum. In general, explanations from all but Adam became clearer. I think that having a written representation to reflect on certainly helped, as did the fact that the representation makes the subset relationships fairly clear. Protocol 3 gives an example of a clear explanation from Justin. He is referring to the drawings in Figure 7. The problem, which he has made notes for, is that Michelle climbs up the ladder 59m, climbs down 88m and then climbs up 37m. The students are supposed to figure out how far she is from her starting place. Before he drew the diagrams, he determined that the answer is 8m up. In Protocol 3, he is explaining just the second sum.

Protocol 3. Justin reasons through 0.

J: I got +8 because I added +29 to that and it made it 0 again. Then from 29 to 37, it is 8m.

T: When you say you added the +29, where’s that coming from, though?  

3 I use size to refer to the unsigned absolute value of a signed quantity.
Further Research

In future papers I hope to provide more detail about each of the students’ constructive trajectories. However, I hope that I have given a flavor of some of the more outstanding challenges students face when first adding and subtracting signed quantities. I am also planning a follow-up study that will look at more students as mathematically strong as Lily to see if they can learn to utilize the notation for addition and subtraction so that situations of combination are addition and MAS situations are subtraction. On the other end of the spectrum, I also plan to work with younger students as they encounter negative quantities for the first time.

One conclusion I can confidently make is that integer instruction is an excellent opportunity for students to move forward their conceptions of number, addition and subtraction. The questions are how they do it, what facilitates it, and what constrains their progress.
SIGNED QUANTITIES

References


